# Order fulfillment: warehouse and inventory models

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# Chapter 4

# Base-stock policies for two-echelon retail inventory systems with lost sales

**Abstract.** *We study a supply chain network consisting of a single warehouse and multiple retailers under periodic review and deterministic lead times where excess demand at the retailers is lost. The retailers face stochastic customer demand and their inventory replenishment orders are satisfied by the warehouse to the extent possible, whereas the warehouse is replenished by an external source with ample supply. The objective is to find replenishment quantities for each stock point such that the long-run average cost per period is minimized, where the total cost consists of unit holding and penalty cost. Besides optimal replenishment policies, we study echelon base-stock policies and show that such policies perform close to optimal (mostly within 1-3% cost increase). However, setting the base-stock levels is non-trivial. The base-stock policy resulting from an equivalent inventory system*

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*with backordered demand (i.e., ignoring lost demand) results in an average cost increase of 4.5-8.5% compared to the best base-stock policy. Therefore, we propose a procedure to decompose the supply chain network when the inventory level at the warehouse is known for a given base-stock policy such that the average holding and penalty cost can be computed separately for each stock point. Expressions for the average satisfied demand at the retailers as well as its distribution are approximated. This decomposition procedure is used to find base-stock levels that minimize the approximated average total cost per period. Numerical experiments indicate that the approximation procedure finds replenishment policies that are mostly within 1% of the best base-stock policy.*

## 4.1. Introduction

In the last decade, the interest in studying inventory systems where excess demand is lost has increased significantly. Especially in many retail settings, customers do not wait for items to be delivered (which is the alternative in systems that assume backordering). Marketing research studies show that stock-out occurrences usually lead to consumers substituting their demand with a different product, brand or sales location (Gruen et al., 2002; Verhoef & Sloot, 2006). In either case, the demand for the original product can be regarded as lost rather than resulting in a delayed purchase (i.e., a backorder). Lost-sales inventory systems are known to be difficult to analyze and the optimal replenishment policy for even single-echelon inventory systems with lost sales is complex when the order delivery lead time is positive (Zipkin, 2008). Alternative replenishment policies are the the main focus in the literature. Some recent work includes studies of base-stock policies (Bijvank & Johansen, 2012; Bijvank et al., 2014), constant-order policies (Goldberg et al., 2016; Xin & Goldberg, 2016) and dual-balancing policies (Levi et al., 2008a). These single-echelon inventory systems ignore that retailers operate in a supply chain network. However, including a multi-echelon setting is considerably more challenging. Replenishment quantities are determined not only by the inventory level of a retailer in question, but also by the inventory levels of the other retailers in the network. In particular, when the supplier (or warehouse) has insufficient inventory available to raise the inventory levels of each retail location to their desired level, it is a challenge to determine appropriate replenishment quantities for each retailer. In this chapter, we capture these dynamics for a basic supply chain composed of a single warehouse and multiple (non-identical) retailers. This supply chain configuration is most common in retail practices (Agrawal & Smith, 2009; Acimovic & Graves, 2017) and, therefore, worthwhile to investigate in more detail.

The warehouse obtains items from an outside supplier that has sufficient stock and replenishes the retailers under periodic review. The transportation (or production) lead times in the system are deterministic. External customer demand arises at the retailers. Demand takes integer values and is assumed to be independent and identically distributed across different time periods. When demand exceeds the on-hand inventory at a retailer, the excess demand is lost and incurs a linear penalty cost. The system incurs linear inventory holding cost for on-hand inventories in all stock points of the supply chain. The planning horizon is infinite, and the objective is to determine a replenishment policy that minimizes the long-run average total costs in the inventory system. This is referred to as the *one-warehouse multi-retailer* (OWMR) problem with stochastic demand in the literature, which we extend to include lost sales in case of excess demand at the retailers. We illustrate the effectiveness of echelon base-stock policies and provide an efficient algorithm to determine good base-stock levels. Under such a policy, each retailer reviews its echelon inventory position at the beginning of every review period and orders a quantity to raise this to a retailer specific base-stock level (defined in more detail in Section 4.3).

The OWMR system has been studied extensively in the literature when retailers have (potentially time-varying) *deterministic demand* and all locations incur a fixed cost for ordering. We refer to Levi et al. (2008b); Shen et al. (2009); Solyah & Süral (2012); Stauffer (2012); Cunha & Melo (2016). In these models, all demand is satisfied. In a random demand environment, Chu & Shen (2010) assume full backordering of unsatisfied demand at the retailers, where a service level constraint is imposed instead of a linear out-of-stock penalty cost in the objective function. The authors propose a power-of-two replenishment policy where items are replenished at constant reorder intervals that are a multiple of a base period. Stenius et al. (2016) include the consolidation of orders from the warehouse to groups of retailers on the same delivery routes.

The majority of the models for the OWMR system (or more general, divergent inventory systems) with *random demand* assume backordering of excess demand and exclude a fixed set-up cost for orders. They mostly focus on analyzing the system with different *inventory rationing policies* where all stock points use base-stock replenishment policies. Such rationing policies determine how the on-hand inventory of the warehouse should be allocated to the retailers in case the sum of the retailers' order sizes exceeds the inventory available at the warehouse. Here, we only include references in case the warehouse is allowed to hold stock (similar to our setting). In the *fair share* (FS) rationing policy, the stock-out probability for the most downstream stock points is equalized (Van Donselaar & Wijngaard, 1987). In the *consistent appropriate share* (CAS) rationing policy the inventory is allocated based on safety stock ratios (De Kok et al., 1994). The *balanced stock* (BS) rationing policy minimizes the expected amount of imbalance, which allows for negative quantities to be allocated to retailers when the warehouse distributes items to the retailers (Van der Heijden et al., 1997). This can be interpreted as allowing lateral transhipments (i.e., shipments between retailers). Diks & De Kok (1998, 1999) and Dogru et al. (2013) develop algorithms to determine (optimal) base-stock levels under the balance assumption (i.e., they assume that the rationing policy always allocates non-negative stock quantities). Van der Heijden (2000) extends this to models subject to service level constraints, rather than a penalty cost for any inventory shortage.

The literature on *multi-echelon inventory systems with lost sales* at the most downstream stage is limited. Paul & Rajendran (2011) use genetic algorithms to find base-stock levels under simple priority allocation rules when the demand is deterministic. McGavin et al. (1993, 1997) and Anupindi & Bassok (1999) were one of the first authors to study the OWMR system with stochastic demand and lost sales. Nahmias & Smith (1994) include a probability that the demand is lost when a customer observes a stockout. These models are simplifications of our model as they assume zero lead times for replenishments. When the material flow to the stock points requires a positive lead time, Huh & Janakiraman (2010b) show for a serial inventory system (i.e., a supply chain where each echelon level consists of only one stock point that replenishes the stock point in the next echelon level downstream) that the order quantities under the optimal policy are a decreasing function of the inventory level and the rate of decrease is between zero and one.

More studies exist for two-echelon inventory systems with lost sales when replenishment orders can be placed under *continuous review*. Andersson & Melchiors (2001) study base-stock policies for such a system where demand is assumed to follow a Poisson process and the lead times are constant. They develop a procedure to set base-stock levels and illustrate the performance for identical retailers. Seifbarghy & Jokar (2006) and Thangam & Uthayakumar (2009) explore fixed order size policies (i.e., $(R, Q)$ policies with reorder level $R$ and fixed order size $Q$) for identical retailers. Hill et al. (2007) extend this to non-identical retailers, but they assume that the lead time of the external supplier to the warehouse is shorter than the lead time of the warehouse to any of the retailers. Haji et al. (2009) introduce a new replenishment policy for the retailers where one unit is ordered each pre-determined time interval. Alvarez & van der Heijden (2014) study base-stock policies where retailers can place emergency orders when the warehouse is out of stock and there is no inventory in transit to the retailer. All these papers consider continuous review and Poisson demand, such that results from queueing theory can be used to analyze the inventory system.

To the best of the authors' knowledge, there have been no studies of divergent inventory systems with periodic reviews and lost sales at the demand-facing locations. Therefore, we formulate two research questions corresponding to the contributions of this chapter: (1) how do base-stock policies perform in comparison to optimal replenishment policies (for moderate to high service levels)? and (2) is there an efficient procedure to set the base-stock levels within the family of base-stock policies? Answering the first question requires us to model the inventory system with optimal replenishment policies as well as with base-stock policies and study the performance. Under the latter policy, retail location $i$ places an order to its upstream echelon to bring its echelon inventory position to level $S_i$ at the beginning of each review period. Base-stock policies are commonly observed in the literature and practice. For the *single-echelon setting*, Huh et al. (2009) and Bijvank et al. (2014) show that such a policy is asymptotically optimal when the lost-sales penalty cost grows large. Bijvank & Johansen (2012) provide an approximation procedure to set the base-stock level. In a *multi-echelon setting*, Acimovic & Graves (2017) study base-stock policies for a system in which demand is fulfilled by other locations of the inventory system in case of a stockout (i.e., lateral transhipments are allowed). Demand is only lost in case there is a system-wide stockout. The only work that considers base-stock policies in a multi-echelon setting where excess demand is lost is for assembly systems (Huh & Janakiraman, 2010a). We numerically compare the performance of base-stock policies to optimal replenishment policies and show that base-stock policies perform close to optimal. In our second contribution, we develop an approximation procedure to analyze the inventory system under study with base-stock policies that can be used to set the base-stock levels. In a numerical study, we illustrate that the approximation procedure finds base-stock levels where the average total cost is within 1% of the performance corresponding to the best base-stock policy.

The outline of the chapter is as follows. In Section 4.2 the notation and assumptions of the problem under study are defined. In Section 4.3 we explore optimal and base-stock policies. Then, in Section 4.4 we derive a procedure to approximate the cost of a given base-stock policy. In Section 4.5, we show that the performance of base-stock policies that are found using the approximated average total cost is close to the best base-stock policies and they significantly outperform base-stock policies for the equivalent inventory problem with backordered demand. Section 4.6 concludes the chapter and offers directions for future research.

## 4.2. Notations and Assumptions

We consider a two-echelon inventory system with one central warehouse and $N$ ($\geq 2$) non-identical retailers as illustrated in Figure 4.1. The retailers are denoted by $1, \ldots, N$ and the warehouse by location 0. Time is divided into intervals of equal length called periods, taken without loss of generality (w.l.o.g.) to be of length one. The time horizon is infinitely long. In every period, the retailers order stock units from the warehouse and the warehouse orders from an external source with unlimited supply. The order arrives at location $i$ after a deterministic lead time of $L_i \in \mathbb{N}$ periods, where $\mathbb{N}$ is the set of positive integers. Consequently, an order released by location $i$ at the beginning of period $t$ will be delivered at location $i$ at the beginning of period $t + L_i$.

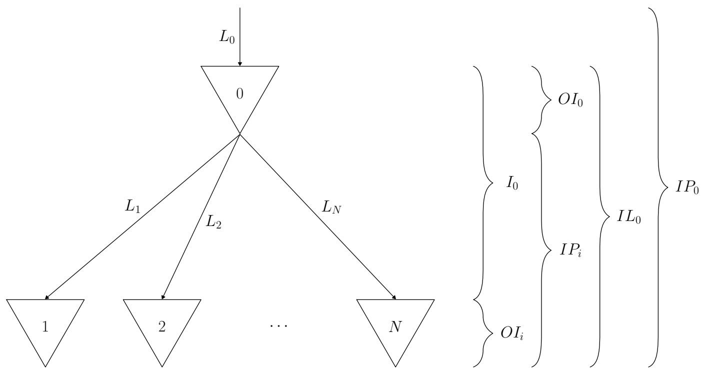


Figure 4.1: The supply chain consists of a central warehouse and $N$ non-identical retailers.

Retailer $i$, $i = 1, 2, \ldots, N$, faces stochastic customer demand, where $\lambda_i$ denotes the mean one-period demand at retailer $i$. We assume the demands in any two different periods to be independent of each other and identically distributed. Let $D_i(t)$ represent the demand at retailer $i$ in period $t$ and $D_i^\tau = \sum_{s=t}^{t+\tau-1} D_i(s)$ the total demand at retailer $i$ over the $\tau$ periods following $t$. The probability and cumulative distribution function of the one-period demand at retailer $i$ is denoted by $g_i(\cdot)$ and $G_i(\cdot)$, respectively. If the demand distribution is over $\tau$ periods, this is indicated by superscript

$\tau$ (that is, $g_i^\tau(\cdot)$ and $G_i^\tau(\cdot)$). We also define

$$\mathcal{G}_i^\tau(s) \quad = \quad \mathbb{E}\big[(s - D_i^\tau)^+\big] = \sum_{d=0}^{s-1} G_i^\tau(d),$$

with $(A)^+ = \max\{A, 0\}$. The customer demand in each period is satisfied to the extent possible, and any demand that cannot be satisfied immediately with on-hand inventory is lost. Demand satisfied at retailer $i$ in the $\tau$ periods starting in period $t$ is denoted $\hat{D}_i^\tau(t)$ and $\hat{D}_i(t) = \hat{D}_i^1(t)$.

We assume the following sequence of events in each period: (i) each stock point $i$, $i = 0, 1, \ldots, N$, receives a shipment ordered $L_i$ periods ago from either the external supplier or the warehouse; (ii) the inventory levels at all stock points are observed and the current period's ordering decisions are made; (iii) the warehouse and external supplier receive the orders and send the shipments; (iv) demand occurs at the retailers during the period where a penalty cost is charged for each customer demand that is lost; (v) holding costs are assessed at the end of the period. The system has the following cost parameters:

$$\begin{aligned} H_i &= \text{installation holding cost per unit per period at stock point } i \\ h_i &= \text{echelon holding cost per unit per period at stock point } i \\ &= H_i - H_0 \text{ for } i = 1, \ldots, N \\ p_i &= \text{lost-sales penalty cost per unit at retailer } i \end{aligned}$$

There is a non-negative added value as materials flow downstream, i.e., $0 \leq H_0 \leq H_i$ for $i = 1, \ldots, N$.

The *installation inventory* at a stock point is defined as the inventory held at and in transit from that location. Thus, the units in the pipeline from the warehouse to the retailers are part of the installation inventory of the warehouse. The *echelon inventory level* is the inventory on hand plus inventories at or in transit to all its downstream stock points. For a retailer, this is just the inventory on hand, whereas the echelon inventory level at the warehouse is the sum of all physical stock in the entire supply chain and all materials that are in transit to the retailers. Note that the installation inventory and echelon inventory level are the same for retailers. The *echelon inventory position* is defined as the echelon inventory level plus the inventory in transit to that stock point. An overview of the notation for the different inventory variables at location $i = 0, 1, \ldots, N$ at time $t$ is included in Table 4.1 as well as Figure 4.1.

Table 4.1: Overview of the notation for the inventory variables

| Notation | Description |
|---|---|
| $OI_i(t)$ | on-hand inventory at stock point $i$ after order delivery at time $t$ (after event (i)), $i = 1, \ldots, N$ |
| $I_i(t)$ | installation inventory at stock point $i$ after order delivery at time $t$ (after event (i))) |
| $IL_i(t)$ | echelon inventory level at stock point $i$ after order delivery at time $t$ (after event (i))) |
| $IP_i^-(t)$ | echelon inventory position at stock point $i$ before a new order is placed at time $t$ (after event (i)) |
| $IP_i^+(t)$ | echelon inventory position at stock point $i$ after a shipment is sent at time $t$ (after event (iii)) |
| $q_i(t)$ | order quantity requested by stock point $i$ at time $t$ (i.e., result of event (ii)) |
| $\hat{q}_i(t)$ | order quantity shipped to stock point $i$ at time $t$ (i.e., result of event (iii)) |

The dynamics of the inventory system are as follows:

$$
\begin{aligned}
IL_i(t) &= I_i(t) = OI_i(t) \\
&= (OI_i(t-1) - D_i(t-1))^+ + \hat{q}_i(t - L_i), \quad i = 1, \ldots, N \\
OI_0(t) &= OI_0(t-1) - \sum_{i=1}^{N} \hat{q}_i(t-1) + q_0(t - L_0) \\
I_0(t) &= OI_0(t) + \sum_{i=1}^{N} \sum_{s=1}^{L_i - 1} \hat{q}_i(t-s) \\
IL_0(t) &= OI_0(t) + \sum_{i=1}^{N} \left[ OI_i(t) + \sum_{s=1}^{L_i-1} \hat{q}_i(t-s) \right] = I_0(t) + \sum_{i=1}^{N} OI_i(t) \\
IP_i^-(t) &= IL_i(t) + \sum_{s=1}^{L_i-1} \hat{q}_i(t-s), \quad i = 0, 1, \ldots, N \\
IP_i^+(t) &= IP_i^-(t) + \hat{q}_i(t), \quad i = 0, 1, \ldots, N
\end{aligned}
$$

Note that the decision quantity $q_i(t)$ at stock point $i$ should be non-negative. Furthermore, the difference between $q_i(t)$ and $\hat{q}_i(t)$ is caused by a shortage at the warehouse, since the warehouse can never ship more than its on-hand inventory. Therefore,

$$\sum_{i=1}^{N} \hat{q}_i(t) = \min\left\{OI_0(t), \sum_{i=1}^{N} q_i(t)\right\} \tag{4.1}$$

$$\hat{q}_i(t) \leq q_i(t), \qquad i = 1, \ldots, N \tag{4.2}$$

Since the supplier of the warehouse has infinite stock, $\hat{q}_0(t) = q_0(t)$.

## 4.3. Replenishment Policies

In this section, we model our two-echelon inventory system according to two different replenishment policies. First, we consider optimal replenishment quantities in Section 4.3.1. Next, we formulate the inventory system where replenishment orders are based on echelon base-stock policies in Section 4.3.2.

### 4.3.1 Optimal Replenishment Policy

Optimal policies for the distribution system described in Section 4.2 are unknown. Replenishment decisions at the beginning of each period are centralized and based on system-wide inventory information, which is assumed to be available and free. As a result, we may assume w.l.o.g. that the retailers never order more than what is available at the warehouse, such that $\hat{q}_i(t) = q_i(t)$ holds for all retailers $i$, $i = 1, \ldots, N$. This means that instead of creating a backlog position at the warehouse, the retailers will attempt to order any difference between the number of units they can order and the number of units they would like to order at the next replenishment epoch.

To find the optimal replenishment policy, we formulate the inventory problem as a Markov Decision Process (MDP). Let vector $\boldsymbol{q}_i \in \mathbb{N}^{L_i-1}$ denote the inventory in transit to stock point $i$ after order delivery at the beginning of a period $t$ (i.e., after event (i)), where the $k$-th component represents the quantity ordered $L_i - k$ periods earlier (i.e., $q_{ik} = q_i(t - (L_i - k))$ for

$k = 1, \ldots, L_i - 1$). The realized stock on hand is denoted by $OI_i$, such that the vector $(OI_i, \boldsymbol{q}_i)$ is the state of stock point $i$ before ordering at event (ii). Hence, the Cartesian product

$$\mathbb{S} \equiv \prod_{i=0}^{N} \left\{ (OI_i, \boldsymbol{q}_i) \mid OI_i \in \mathbb{N}, \boldsymbol{q}_i \in \mathbb{N}^{L_i - 1} \right\}$$

describes the state space of the MDP. For a given state $\xi_t \in \mathbb{S}$ at period $t$, the actions $\boldsymbol{q}(\xi_t) = (q_0(t), q_1(t), \ldots, q_N(t))$ represent the number of units ordered by stock point $i$, $i = 0, 1, \ldots, N$, and need to satisfy

$$\sum_{i=1}^{N} q_i(t) \leq OI_0.$$

Let $y_i$ represent the realized installation inventory and $z_i$ the realized echelon inventory level at stock point $i$ after ordering and shipping (i.e., after event (iii)). Consequently, $y_i(\xi_t) = z_i(\xi_t) = OI_i$ for retailer $i$, $i = 1, \ldots, N$, and

$$y_0(\xi_t) = \left( OI_0 - \sum_{i=1}^{N} q_i(t) \right) + \sum_{i=1}^{N} \left[ \sum_{k=1}^{L_i - 1} q_{ik} + q_i(t) \right] = OI_0 + \sum_{i=1}^{N} \sum_{k=1}^{L_i - 1} q_{ik},$$

which is the stock on hand at the warehouse plus all stock in transit downstream of the warehouse (i.e., to the retailers), and

$$z_0(\xi_t) = y_0(\xi_t) + \sum_{i=1}^{N} OI_i.$$

For simplicity, rewrite $y_i(\xi_t)$ and $z_i(\xi_t)$ as $y_i$ and $z_i$, respectively. The direct one-period costs are given by the sum of expected holding and penalty cost at all locations

$$c(\xi_t) = H_0 y_0 + \sum_{i=1}^{N} H_i \mathbb{E}\left[ (y_i - D_i(t))^+ \right] + \sum_{i=1}^{N} p_i \mathbb{E}\left[ (D_i(t) - y_i)^+ \right],$$

or equivalently

$$c(\xi_t) = h_0 \left( z_0 - \sum_{i=1}^{N} \mathbb{E}\left[\min\{D_i(t), z_i\}\right] \right) + \sum_{i=1}^{N} h_i \mathbb{E}\left[ (z_i - D_i(t))^+ \right] + \sum_{i=1}^{N} p_i \mathbb{E}\left[ (D_i(t) - z_i)^+ \right].$$

See Appendix 4.A for more details.

Given the state space, the direct costs and the transition probabilities as implied by the demand distribution, we can formulate the Bellmann equation as

$$V_{t+1}(\xi_t) = c(\xi_t) + \min_{\boldsymbol{q}(\xi_t) | \sum_{i=1}^{N} q_i(t) \leq OI_0} \left\{ \mathbb{E}\left[ V_t \left( \left( OI_0 - \sum_{i=1}^{N} q_i(t) + q_{0,1}, \mathbf{F}(\boldsymbol{q}_0), q_0(t) \right), \prod_{i=1}^{N} \left( (OI_i - D_i(t))^+ + q_{i,1}, \mathbf{F}(\boldsymbol{q}_i), q_i(t) \right) \right) \right] \right\},$$

where $\mathbf{F}(\boldsymbol{q})$ is a function that removes the first element of the vector $\boldsymbol{q}$. The optimal policy and the corresponding average total costs can be found by recursively solving the above equation with value iteration (Puterman, 2005).

### 4.3.2 Base-Stock Policy

As mentioned in Section 4.1, a relevant class of ordering policies is constituted by the class of echelon base-stock policies. In our multi-echelon setting, such a policy is denoted by the vector $\boldsymbol{S} = (S_0, \ldots, S_N)$, where $S_i \in \mathbb{N}$ denotes the desired base-stock level for the echelon inventory position at location $i$. Thus, $q_i(t) = S_i - IP_i^-(t)$. For retailer $i$, $i = 1, \ldots, N$,

$$q_i(t) = S_i - \left( OI_i + \sum_{k=1}^{L_i - 1} \hat{q}_{ik} \right),$$

while for the warehouse

$$\begin{aligned} q_0(t) &= S_0 - \left(OI_0 - \sum_{i=1}^{N} \hat{q}_i(t) + \sum_{k=1}^{L_0-1} q_{0k}\right) - \sum_{i=1}^{N}\left(OI_i + \sum_{k=1}^{L_i-1} \hat{q}_{ik} + \hat{q}_i(t)\right) \\ &= S_0 - \left(OI_0 + \sum_{k=1}^{L_0-1} q_{0k}\right) - \sum_{i=1}^{N}\left(OI_i + \sum_{k=1}^{L_i-1} \hat{q}_{ik}\right). \end{aligned}$$

The number of units shipped to the warehouse always equals the quantity ordered (i.e., $\hat{q}_0(t) = q_0(t)$). However, the actual amount that is shipped to retailer $i$ (i.e., $\hat{q}_i(t)$) for $i = 1, \ldots, N$, depends on the on-hand inventory at the upstream location (i.e., location 0). Define the shortage at the warehouse in period $t$ as

$$\Omega_0(t) = \left(\sum_{i=1}^{N} q_i(t) - OI_0(t)\right)^+ = \left(\sum_{i=1}^{N} S_i - IL_0(t)\right)^+. \tag{4.3}$$

When $\Omega_0(t) = 0$, there is no shortage at the warehouse and $\hat{q}(t) = q(t)$. Otherwise, there is insufficient stock at the warehouse to satisfy the replenishment orders of all retailers and the on-hand inventory at the warehouse needs to be allocated to the retailers with the use of rationing functions $r_i(\xi_t)$. In this chapter, we assume $r(\cdot)$ to be a linear allocation function (Diks & De Kok, 1999). This implies that all retailers are allocated a fixed fraction $f_i$ of the shortage at the warehouse. Hence

$$r_i(\xi_t) = q_i(t) - f_i\Omega_0(t). \tag{4.4}$$

If the allocated shortage is larger than the amount originally ordered, i.e. $q_i(t) < f_i\Omega_0(t)$, the rationing quantity $r_i(\xi_t)$ will be negative. In order to facilitate our analysis, we assume that such situations do not happen and $\hat{q}_i = r_i(\xi_t)$ for all locations $i, i = 1, \ldots, N$. This assumption is referred to as the *balance assumption*, since it implies that the inventory is sufficiently balanced over the retailers (Van der Heijden et al., 1997; Diks & De Kok, 1998).

Note that any allocated shortages will not be backordered at the warehouse. If there is a shortage at the warehouse, the retailers will attempt to order that difference at the next replenishment epoch. Consider the dynamics in the following example.

**Example 4.1.** *Consider a supply chain that consists of a warehouse with base-stock level* $S_0 = 5$ *and two retailers each with base-stock level* $S_i = 2$, $i = 1, 2$. *The lead time* $L_i = 1$ *for* $i = 0, 1, 2$. *After order delivery at time period* $t$, *we observe* $OI_0 = OI_1 = OI_2 = 1$ *and no orders are outstanding. The retailers order one unit each (i.e.,* $q_i(t) = 1$ *for* $i = 1, 2$*). Since there is a shortage at the warehouse, there is only one unit shipped to retailer 1 and nothing to retailer 2 (i.e.,* $\hat{q}_1(t) = 1$ *and* $\hat{q}_2(t) = 0$*). The echelon inventory position at the retailers after ordering and shipping becomes* $IP_1^+(t) = 2$ *and* $IP_2^+(t) = 1$, *whereas the echelon inventory position at the warehouse before ordering becomes* $IP_0^-(t) = 3$ *such that the warehouse orders* 2 *units. When these units arrive to the warehouse at the beginning of period* $t + 1$, *retailer 2 will order at least one unit such that the warehouse will never have more than* $S_0 - \sum_{i=1}^{N} S_i$ *units available on hand during a period. Consequently, it becomes clear that* $S_0 \geq S_1 + S_2$.

Furthermore, note that the rationing function in Eq. (4.4) can result in fractional quantities. Since we consider discrete demand, we round the quantities $\hat{q}_i$ up for the first retailers and down for the other retailers such that $\sum_{i \geq 1}(q_i(t) - \hat{q}_i(t)) = \Omega_0(t)$.

## 4.4. Cost Approximation for Base-Stock Policies

In this section, we develop a heuristic procedure to approximate the long-run average total costs of the two-echelon divergent inventory system with echelon base-stock replenishment policies where the vector of base-stock levels $\boldsymbol{S} = (S_0, S_1, \ldots, S_N)$ is given. Our approximation decomposes the costs of the total inventory system into the costs for each retailer and the warehouse, such that the expected total costs of the inventory system is calculated as

$$h_0 \mathbb{E}[IL_0] + \sum_{i=1}^{N} h_i \mathbb{E}\left[(IL_i - D_i)^+\right] + \sum_{i=1}^{N} p_i \mathbb{E}\left[(D_i - IL_i)^+\right]. \tag{4.5}$$

This approach requires an analysis of the steady-state behavior for the echelon inventory level $IL_i$ at each inventory location $i = 0, 1, \ldots, N$.

Let us first study the retailers. The probability distribution function for $IL_i(t)$, $i = 1, \ldots, N$, is given by

$$P\left(IL_i(t) = x_i\right) = P\left(IP_i^+(t - L_i) - \sum_{s=0}^{L_i - 1} q_i(t-s) = x_i\right), \tag{4.6}$$

since the inventory level increases during the time interval $[t - L_i, t]$ due to the outstanding orders (after ordering) at time $t - L_i$ (which is captured by $IP_i^+(t - L_i)$) and it decreases due to the satisfied demand during the previous $L_i$ time periods (which is captured by the amounts ordered $q_i$).

Note that $IP^+(t) = S_i - [q_i(t) - \hat{q}_i(t)]$ since the warehouse can have a shortage. For ease of notation, let's define $\overline{S}_N = \sum_{i=1}^{N} S_i$. Under the balance assumption, Eq. (4.4) defines $\hat{q}_i(t)$ such that $IP_i^+(t) = S_i - f_i \Omega_0(t)$, where $\Omega_0(t) = \left(\overline{S}_N - IL_0(t)\right)^+$ according to Eq. (4.3). When we condition

$IP_i^+$ on $IL_0$, this results in

$$P(IP_i^+(t) = y) = \sum_{x_0=0}^{S_0} P\left(IP_i^+(t) = y \mid IL_0(t) = x_0\right) P(IL_0 = x_0) \quad (4.7)$$

$$= \begin{cases} \sum_{x_0=\overline{S}_N}^{S_0} P(IL_0(t) = x_0) & \text{if } y = S_i \\ P\left(IL_0(t) = \overline{S}_N - \frac{S_i - y}{f_i}\right) & \text{if } 0 \le y < S_i \\ 0 & \text{otherwise} \end{cases} \quad (4.8)$$

since $IP_i^+(t) = S_i - f_i\left(\overline{S}_N - x_0\right)^+$ when $IL_0(t) = x_0$. This requires a probability distribution function for $IL_0$. Furthermore, note that the ratio in the second condition of Eq. (4.8) can require rounding as indicated at the end of Section 4.3.2.

The second random variable in Eq. (4.6) is the satisfied demand at retailer $i$ over the time interval $[t-L_i, t]$. We will approximate this component as a function of $IL_0(t - L_i)$ and denote this by $\hat{D}_i^{L_i}(IL_0)$, since the availability of the item at the retailer depends on the pipeline inventory at and downstream of the warehouse. This implies that $\sum_{s=0}^{L_i-1} q_i(t - s)$ (as used in Eq. (4.6)) is represented by the random variable $\hat{D}_i^{L_i}(IL_0)$ such that $P(\hat{D}_i^{L_i}(x_0) = d) = P(\hat{D}_i^{L_i} = d \mid IL_0 = x_0)$.

The probability distribution function of the satisfied demand $\hat{D}_i^{L_i}$ over $L_i$ time periods is dependent on the entire state description since it depends on the individual order quantities delivered during this time interval (as discussed in Section 4.3). For the single-echelon system, Bijvank & Johansen (2012) use the demand distribution $g_i^{L_i}$ in combination with a correction factor to correct for any out-of-stock occurrences during the lead time $L_i$. A numerical study by Cardos et al. (2017) shows that this approach can result in negative probabilities when the service level is low (which corresponds to a scenario where the base-stock level is relatively low compared to the average demand). Therefore, a different approach is taken in this chapter. We rescale the actual demand distribution $g_i^{L_i}$ and denote the new distribution by $\hat{g}_i^{L_i}$ to represent the probability distribution function for $\hat{D}_i^{L_i}$. One can imagine that the expectation of the satisfied demand $\hat{D}_i^{L_i}$ is smaller than the expectation of the actual demand $D_i^{L_i}$. We will use the following notation:

$$\mathbb{E}\left[\hat{D}_i^{L_i}(x_0)\right] = \mathbb{E}\left[\hat{D}_i^{L_i} \mid IL_0 = x_0\right] = \mu_i(x_0) \leq \lambda_i L_i = \mathbb{E}\left[D_i^{L_i}\right]. \quad (4.9)$$

The expectation over the conditional expectations $\mu_i(IL_0)$ represents the expected total satisfied demand over $L_i$ time periods. This can be directly related to the fill rate $\beta_i$ realized at retailer $i$, where the fill rate is defined as the fraction of the demand immediately satisfied by inventory on hand. In particular,

$$\mathbb{E}\left[\mu_i(IL_0)\right] = \sum_{x_0=0}^{S_0} \mu_i(x_0) P(IL_0 = x_0) = \beta_i \lambda_i L_i. \quad (4.10)$$

Based on simulation experiments, we identified the following functional form to approximate $\mu_i(x_0)$:

$$\tilde{\mu}_i(x_0) = \left(a_i \left(S_0 - x_0\right)^2 + \alpha_i \lambda_i L_i\right)^+, \quad (4.11)$$

where $a_i$ and $\alpha_i$ are scalar parameters for retailer $i$. Figure 4.2 illustrates the actual expectations $\mu_i(x_0)$ for six different numerical instances that are found with simulation. For this figure we selected instances with a moderate average fill rate (90-92%) such that lost-sales occurrences frequently occur. The first two subfigures are for the first retailer only since the two retailers are symmetrical, whereas the third and fourth subfigures are for the first and second retailer respectively since the parameter setting is asymmetrical (the same for the last two subfigures). Note that $\tilde{\mu}_i(x_0) = 0$ for $x_0 = 0, \ldots, x_0^* - 1$ and $\tilde{\mu}_i(x_0) > 0$ for $x_0 = x_0^*, \ldots, S_0$ where

$$x_0^* = \left\lceil S_0 - \sqrt{\frac{-\alpha_i \lambda_i L_i}{a_i}} \right\rceil \quad (4.12)$$

and $\lceil x \rceil$ is $x$ rounded up to the nearest integer.

The value of $a_i$ can be found by using the approximation of Eq. (4.11)

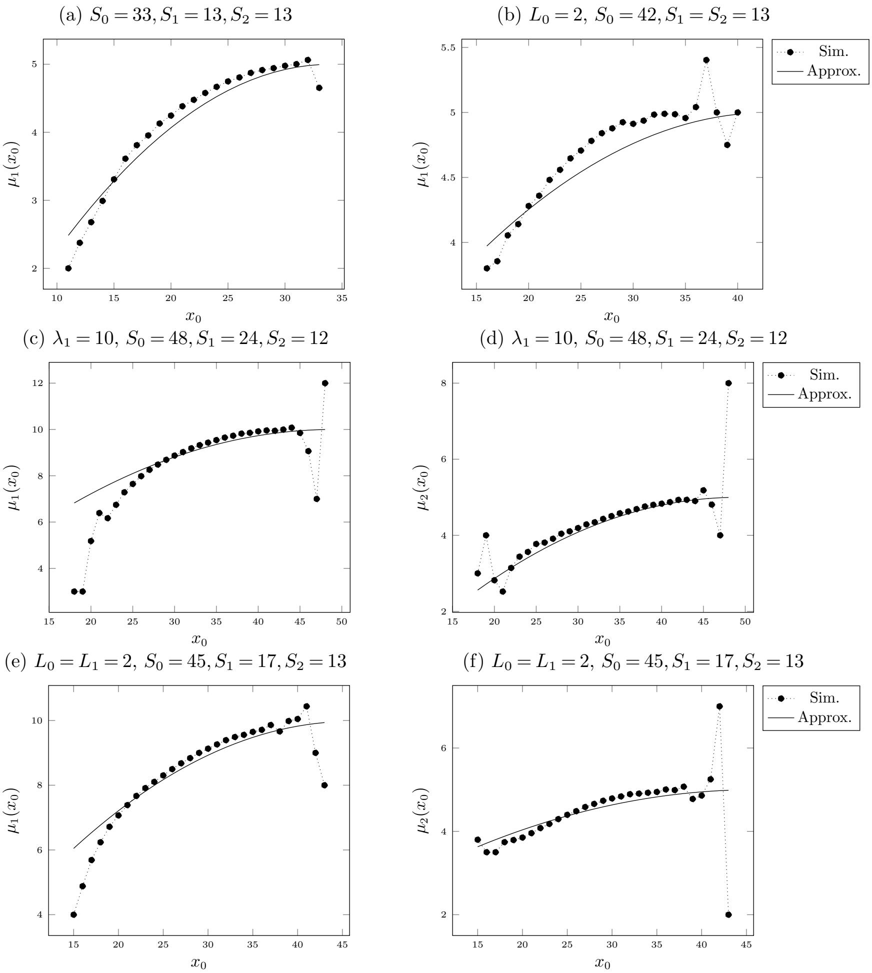


Figure 4.2: Simulated mean lead time demands $\mu_i(x_0)$ (Sim.) and its approximation $\tilde{\mu}_i(x_0)$ (Approx.) for an OWMR system with two retailers. The lead times are $L_0 = L_1 = L_2 = 1$, whereas demand follows a Poisson distribution with averages $\lambda_1 = \lambda_2 = 5$, unless indicated otherwise. The selected base-stock levels are those that minimize the average total holding cost within the family of base-stock policies where $h_i = 1$ for $i = 0, 1, 2$ and $p_1 = p_2 = 9$.

in Eq. (4.10) when the fill rate $\beta_i$ is known:

$$\begin{aligned} \mathbb{E}\left[\mu_i(IL_0)\right] &= \beta_i \lambda_i L_i = \textstyle\sum_{x_0=0}^{S_0} P(IL_0 = x_0)\mu_i(x_0) \\ &\approx \textstyle\sum_{x_0=x_0^*}^{S_0} P(IL_0 = x_0)\left[a_i\left(S_0 - x_0\right)^2 + \alpha_i \lambda_i L_i\right] \end{aligned} \tag{4.13}$$

such that

$$a_i = \frac{\lambda_i L_i \left(\beta_i - \sum_{x_0=x_0^*}^{S_0} P(IL_0 = x_0)\alpha_i\right)}{\sum_{x_0=x_0^*}^{S_0} P(IL_0 = x_0)\left(S_0 - x_0\right)^2}. \tag{4.14}$$

The value of $\alpha_i \lambda_i L_i$ represents an upper bound on $\mu_i(IL_0)$ (i.e., the maximum expected demand to be satisfied by retailer $i$ during $L_i$ time periods). To define $\alpha_i$, consider the scenario where $OH_i(t) = S_i$ such that the inventory level at retailer $i$ is at the maximum at time $t$ and no order is outstanding. This corresponds to the case in which retailer $i$ is able to satisfy most demand. Consequently, we define $\alpha_i \lambda_i L_i = \mathbb{E}\left[\min\{D_i^{L_i}, S_i\}\right]$ such that

$$\begin{aligned} \alpha_i &= \frac{\mathbb{E}\left[\min\{D_i^{L_i}, S_i\}\right]}{\lambda_i L_i} = \frac{\mathbb{E}[D_i^{L_i}] - \mathbb{E}\left[\left(D_i^{L_i} - S_i\right)\right]}{\lambda_i L_i} && (4.15) \\ &= \frac{S_i - \mathbb{E}\left[(S_i - D_i^{L_i})^+\right]}{\lambda_i L_i} = \frac{S_i - \mathcal{G}_i^{L_i}(S_i)}{\lambda_i L_i}. && (4.16) \end{aligned}$$

Note that $\alpha_i \geq \beta_i$ such that $a_i \leq 0$. Furthermore, note that $a_i$ depends on $x_0^*$ in Eq. (4.14) and $x_0^*$ depends on $a_i$ in Eq. (4.12). Therefore, we need an iterative procedure where we start with $x_0^* = 0$ until the values of $a_i$ and $x_0^*$ do not change anymore. This is illustrated in Algorithm 1.

**Algorithm 1** Determine $a_i$ and $\alpha_i$ for a given distribution $P(IL_0 = x_0)$ and fill rate $\beta_i$

1: Set $x_0^* = 0$ and calculate $\alpha_i$ with Eq. (4.16)
2: Calculate $a_i$ according to Eq. (4.14) and set

$$\tilde{x}_0^* = \left\lceil S_0 - \sqrt{\frac{-\alpha_i \lambda_i L_i}{a_i}} \right\rceil$$

3: If $\tilde{x}_0^* = x_0^*$, then go to Step 4; otherwise set $x_0^* = \tilde{x}_0^*$ and go to Step 2
4: Terminate the algorithm

Let's go back to rescaling $g_i^{L_i}$ to create $\hat{g}_i^{L_i}$ (i.e., the probability distribution function of the approximated satisfied demand over $L_i$ time periods when the inventory level at the warehouse (i.e., $IL_0 = x_0$) is known), where we now have a procedure to approximate the expectation for $\hat{D}_i^{L_i}$ when fill rate $\beta_i$ and a probability distribution of $IL_0$ are known. As indicated in

Eq. (4.9), this expectation will be smaller than the expectation for $D_i^{L_i}$. Therefore, we first truncate the domain of $g_i^{L_i}$ at $\overline{M}$ such that the normalized expectation over $d = 0, \ldots, \overline{M}$ exceeds $\tilde{\mu}_i(x_0)$ for the first time:

$$\overline{M} = \min \left\{ M \,\middle|\, \frac{\sum_{d=0}^{M} d g_i^{L_i}(d)}{\sum_{d=0}^{M} g_i^{L_i}(d)} \geq \tilde{\mu}_i(x_0) \right\}. \tag{4.17}$$

Since the expectation exceeds $\tilde{\mu}_i(x_0)$, we need to redistribute the probability mass such that the expectation over the satisfied demand equals $\tilde{\mu}_i(x_0)$. Hence, we define

$$\hat{g}_i^{L_i}(d) = \begin{cases} c g_i^{L_i}(d) & \text{if } 0 \leq d \leq \overline{M} - 1 \\ 1 - c G_i^{L_i}(d-1) & \text{if } d = \overline{M} \\ 0 & \text{otherwise} \end{cases} \tag{4.18}$$

where

$$c = \frac{\overline{M} - \tilde{\mu}_i(x_0)}{\overline{M} G_i^{L_i}(\overline{M} - 1) - \sum_{d=0}^{\overline{M}-1} d g_i^{L_i}(d)}. \tag{4.19}$$

We can use this distribution to directly approximate the probability distribution of the on-hand inventory level at retailer $i$ as $P(OI_i = x_i) = \hat{g}_i^{L_i}(S - x_i)$. Consequently, the satisfied demand at retailer $i$ during a single review period is given by

$$P(\hat{D}_i = d) = P(D_i = d)P(OI_i \geq d) + P(D_i > d)P(OI_i = d), \tag{4.20}$$

and the fill rate at retailer $i$ is approximated as

$$\beta_i = \frac{\sum_d d P(\hat{D}_i = d)}{\lambda_i}. \tag{4.21}$$

Next, we consider the total satisfied customer demand in a single review period for the entire supply chain, denoted as $\hat{D}$. To obtain the probability distribution of the satisfied demand $\hat{D}$ over all retail locations, we take the convolution of $\hat{D}_i$ over all retailers $i = 1, \ldots, N$. We can use this to specify the probability distribution of the inventory level at the warehouse as

$$P(IL_0 = x_0) = P\big((\hat{D})^{*L_0} = S_0 - x_0\big), \tag{4.22}$$

where $(\hat{D})^{*L_0}$ is the $L_0$-fold convolution with $\hat{D}$ itself.

In summary, Eq. (4.21) and (4.22) can be found with Eq. (4.18)-(4.20) and require an expectation for $\hat{D}_i^{L_i}(x_0)$, which is given by Eq. (4.11) and Algorithm 1 defines the values of $a_i$ and $\alpha_i$. This random variable $\hat{D}_i^{L_i}(x_0)$ is used for the order quantities in Eq. (4.6) together with the distribution specified in Eq. (4.22) that is used in Eq. (4.8) and eventually in Eq. (4.6) as well. The distributions in Eq. (4.6) and (4.22) are used to calculate the average total system-wide cost in Eq. (4.5). A more detailed summary of the approximation procedure is provided in Algorithm 2.

**Algorithm 2** Determine the probability distribution of $IL_i$ for all stock points $i = 0, 1, \ldots, N$ as well as the average total cost

1: Set iteration $n = 0$, where we initialize $\beta_i^{(n)} = 0.9$ for all retailers $i = 1, \ldots, N$ and

$$P^{(n)}(IL_0 = x_0) = P\left(\sum_{i=1}^{N}(D_i)^{*L_0} = S_0 - x_0\right)$$

2: **repeat**
3: Set $n = n + 1$
4: **for** $i = 1, \ldots, N$ **do**
5: Use Algorithm 1 to find $a_i$ and $\alpha_i$
6: **for** $x_0 = 0, \ldots, S_0$ **do**
7: Calculate $\tilde{\mu}_i(x_0)$ with Eq. (4.11) and $\overline{M}$ with Eq. (4.17).
8: Define $P(OI_i = x_i) = \hat{g}_i^{L_i}(S_i - x_i)$ where $\hat{g}_i^{L_i}(d)$ is specified by Eq. (4.18).
9: Use Eq. (4.20) and Eq. (4.21) to define $P(\hat{D}_i = d)$ and $\beta_i(x_0)$, respectively
10: **end for**
11: Calculate $\beta_i = \sum_{x_0=0}^{S_0} P^{(n-1)}(IL_0 = x_0)\beta_i(x_0)$
12: Update

$$\beta_i^{(n)} = \frac{1}{2}\left(\beta_i^{(n-1)} + \beta_i\right)$$

13: **end for**
14: Take the convolution of $\hat{D}_i$ over all retailers $i = 1, \ldots, N$ to obtain the probability distribution for $\hat{D}$, where the distribution of $\hat{D}_i$ is given by Eq. (4.20)
15: Take the $L_0$-convolution of $\hat{D}$ with itself to obtain the probability distribution for $(\hat{D})^{*L_0}$
16: Update $P^{(n-1)}(IL_0 = x_0)$ to $P^{(n)}(IL_0 = x_0)$ with Eq. (4.22)
17: **until** $\mid \beta_i^{(n)} - \beta_i^{(n-1)} \mid < \epsilon$ for all $i = 1, \ldots, N$
18: The probability distribution of $IP_i^+$ is given by Eq. (4.8) and $P^{(n)}(IL_0 = x_0)$ above
19: The probability distribution of $\sum_{s=0}^{L_i-1} q_i(t-s)$ is given by $P(\hat{D}_i^{L_i} = d \mid IL_0 = x_0)$ and Eq. (4.18)-(4.19)
20: Use these distributions in Eq. (4.6) to find the expected total cost in Eq. (4.5)

## 4.5. Numerical results

The aim of this section is to illustrate the performance of echelon base-stock policies relative to optimal replenishment policies as well as the performance of our approximation procedure. In our numerical study, we also include the performance of base-stock policies resulting from backorder models that exist in the literature. For the set-up of our numerical instances, we use Zipkin (2008) as guideline and extended this to a two-echelon setting. In particular, we consider a supply chain consisting of one warehouse and two retailers. Because the Markov chain grows exponentially in the number of retailer locations as well as the lead time (see Section 4.3), we consider only lead times of either one or two review periods. In particular, we study the following three supply chain configurations: $(L_0, L_1, L_2) \in \{(1,1,1), (2,1,1), (2,2,1)\}$. The latter configuration already results in an MDP where the state space has five dimensions (as described in Section 4.3). The demand at retailer $i$, $i = 1, 2$, follows a Poisson distribution with an average of 5 or 10, and the unit penalty cost varies in $p = 4, 9, 19, 39$. Without loss of generality, the unit holding cost is set to one (i.e., $h_i = 1$ for $i = 0, 1, 2$) and the length of the review period is also set to one. Results for instances where the supply chain network consists of three retail locations (i.e., configuration $(L_0, L_1, L_2, L_3) = (1, 1, 1, 1)$) or where the demand follows a geometric distribution, are included in Appendix 4.B, since the observations from these instances are similar to the results of a supply chain network with two retailers and Poisson distributed demand. This is in line with the work by Dogru et al. (2009) that indicates that the number of retailers does not influence the results when the balance assumption is used in a multi-echelon inventory system.

The detailed results are reported in Table 4.2 to 4.4 for each supply chain configuration respectively. Column 1 and 2 present the parameter settings for the average demand and unit penalty cost in each instance. In columns 3 to 6, we report the base-stock levels $\boldsymbol{S}^*$ of the best base-stock policy, the corresponding average total cost $C(\boldsymbol{S}^*)$ as well as the average fill rate for the two retailers (denoted by $\beta_i(\boldsymbol{S}^*)$ for retailer $i = 1, 2$, respectively). In column 7, we report the relative cost increase for using the best base-stock policy with the balance assumption where $f_i = S_i/\overline{S}_N$ compared to the optimal policy:

$$CI(*) = \frac{C(\boldsymbol{S}^*) - C(*)}{C(*)}, \tag{4.23}$$

where $C(*)$ is the average total cost associated with the optimal policy. Columns 8 and 9 report the best base-stock levels $\boldsymbol{S}^{BO}$ of the equivalent inventory system with the backorder assumption in case there is a stockout as well as the relative cost increase for using this base-stock policy compared to the best base-stock policy. This cost increase for a certain base-stock policy $\boldsymbol{S}$ is given by

$$CI(\boldsymbol{S}) = \frac{C(\boldsymbol{S}) - C(\boldsymbol{S}^*)}{C(\boldsymbol{S}^*)}, \tag{4.24}$$

where $C(\boldsymbol{S})$ is the average total cost associated with base-stock policy $\boldsymbol{S}$. Similarly, columns 10 and 11 report the base-stock levels $\overline{\boldsymbol{S}}$ found with our approximation procedure and the relative cost increase for using this policy compared to the best base-stock policy.

Note that the results for the third supply chain configuration in Table 4.4 (i.e., when $(L_0, L_1, L_2) = (2, 2, 1)$) are found with simulation when $\lambda_i = 10$ for $i = 1$ or 2 due to storage space limitations (these instances require a state description in five dimensions). Furthermore, no optimal replenishment policies could be found for these instances for the same reason. A summary of all results is provided in Figure 4.3.

Table 4.2: Results of the supply chain configuration where $L_0 = 1$ and $L_1 = L_2 = 1$.

| scenario | | best base-stock policy | | | | | backorder model | | approximation | |
|---|---|---|---|---|---|---|---|---|---|---|
| $(\lambda_1, \lambda_2)$ | $(p_1, p_2)$ | $\boldsymbol{S}^*$ | $C(\boldsymbol{S}^*)$ | $\beta_1(\boldsymbol{S}^*)$ | $\beta_2(\boldsymbol{S}^*)$ | $CI(*)$ | $\boldsymbol{S}^{BO}$ | $CI(\boldsymbol{S}^{BO})$ | $\overline{\boldsymbol{S}}$ | $CI(\overline{\boldsymbol{S}})$ |
| 5,5 | 4,4 | 10,10,26 | 20.95 | 80.09% | 77.38% | 3.31% | 13,13,33 | 11.34% | 13,13,26 | 0.16% |
| 5,5 | 9,9 | 13,13,33 | 27.55 | 92.29% | 90.81% | 2.42% | 14,14,37 | 5.99% | 14,14,33 | 0.28% |
| 5,5 | 19,19 | 15,14,37 | 33.50 | 96.39% | 95.29% | 1.52% | 16,16,39 | 1.81% | 15,15,36 | 0.56% |
| 5,5 | 39,39 | 16,16,41 | 38.84 | 98.45% | 98.08% | 0.91% | 17,17,42 | 1.05% | 16,16,41 | 0.00% |
| 5,5 | 4,9 | 10,13,29 | 24.24 | 78.32% | 90.50% | 2.88% | 13,14,35 | 8.68% | 12,15,29 | 0.39% |
| 5,5 | 4,19 | 10,15,32 | 27.25 | 79.98% | 96.19% | 2.61% | 13,16,36 | 5.34% | 11,16,32 | 0.36% |
| 5,5 | 4,39 | 10,16,34 | 29.98 | 81.42% | 98.08% | 2.49% | 13,17,38 | 5.64% | 12,17,35 | 1.72% |
| 5,5 | 9,19 | 13,15,35 | 30.51 | 91.84% | 95.78% | 1.91% | 14,16,38 | 3.56% | 13,15,35 | 0.00% |
| 5,5 | 9,39 | 12,16,37 | 33.25 | 90.86% | 98.30% | 1.81% | 14,17,40 | 3.60% | 13,17,37 | 0.09% |
| 5,5 | 19,39 | 14,16,39 | 36.14 | 95.88% | 98.17% | 1.10% | 16,17,41 | 2.38% | 14,16,37 | 1.58% |
| 10,5 | 4,4 | 21,10,41 | 28.27 | 86.45% | 79.35% | 2.97% | 24,13,49 | 10.81% | 24,12,41 | 0.27% |
| 10,5 | 9,9 | 24,12,48 | 36.19 | 93.87% | 89.80% | 2.27% | 26,14,53 | 4.54% | 27,14,48 | 0.31% |
| 10,5 | 19,19 | 26,14,54 | 43.25 | 97.21% | 96.09% | 1.43% | 28,16,57 | 3.00% | 30,16,53 | 0.51% |
| 10,5 | 39,39 | 29,16,58 | 49.58 | 98.75% | 98.27% | 0.88% | 29,17,60 | 1.38% | 29,16,58 | 0.00% |
| 10,5 | 4,9 | 21,13,44 | 31.51 | 85.97% | 91.33% | 2.57% | 24,14,51 | 8.54% | 23,14,44 | 0.26% |
| 10,5 | 4,19 | 21,15,46 | 34.45 | 86.00% | 95.81% | 2.30% | 24,16,52 | 6.11% | 22,15,48 | 0.56% |
| 10,5 | 4,39 | 20,16,48 | 37.12 | 85.79% | 98.22% | 2.16% | 24,17,54 | 6.19% | 22,17,49 | 0.25% |
| 10,5 | 9,19 | 24,15,51 | 39.10 | 93.80% | 96.74% | 1.86% | 26,16,55 | 4.49% | 24,15,51 | 0.00% |
| 10,5 | 9,39 | 24,16,52 | 41.70 | 93.79% | 97.96% | 1.52% | 26,17,56 | 3.22% | 24,16,52 | 0.00% |
| 10,5 | 19,39 | 26,16,55 | 45.83 | 96.84% | 98.19% | 1.08% | 28,17,58 | 1.91% | 26,17,54 | 1.28% |
| 10,5 | 9,4 | 24,10,45 | 32.99 | 93.20% | 80.39% | 2.85% | 26,13,51 | 6.16% | 26,11,45 | 0.28% |
| 10,5 | 19,4 | 26,10,49 | 37.18 | 97.08% | 82.05% | 2.59% | 28,13,53 | 4.32% | 28,11,50 | 0.62% |
| 10,5 | 39,4 | 29,10,51 | 40.99 | 98.65% | 80.82% | 2.60% | 29,13,55 | 3.91% | 29,10,49 | 1.94% |
| 10,5 | 19,9 | 26,12,51 | 40.34 | 96.93% | 90.32% | 1.85% | 28,14,56 | 4.40% | 27,13,52 | 0.08% |
| 10,5 | 39,9 | 28,12,54 | 44.11 | 98.64% | 90.94% | 1.72% | 29,14,57 | 2.14% | 30,13,54 | 0.47% |
| 10,5 | 39,19 | 28,14,56 | 46.98 | 98.57% | 96.00% | 1.20% | 29,16,59 | 2.60% | 29,14,56 | 0.04% |
| 10,10 | 4,4 | 21,20,55 | 35.59 | 86.48% | 84.82% | 2.78% | 24,24,64 | 8.39% | 23,23,55 | 0.15% |
| 10,10 | 9,9 | 24,24,64 | 44.72 | 94.24% | 93.48% | 2.00% | 26,26,69 | 3.90% | 25,25,64 | 0.15% |
| 10,10 | 19,19 | 26,26,70 | 52.88 | 97.31% | 96.92% | 1.27% | 28,28,73 | 1.68% | 28,27,69 | 0.46% |
| 10,10 | 39,39 | 28,28,75 | 60.17 | 98.79% | 98.60% | 0.73% | 29,29,77 | 0.67% | 28,28,75 | 0.00% |
| 10,10 | 4,9 | 20,24,59 | 40.16 | 84.96% | 93.59% | 2.41% | 24,26,67 | 6.73% | 22,26,61 | 0.45% |
| 10,10 | 4,19 | 20,27,63 | 44.29 | 85.64% | 97.53% | 2.24% | 24,28,69 | 4.94% | 21,27,63 | 0.09% |
| 10,10 | 4,39 | 20,29,65 | 48.01 | 85.76% | 98.70% | 2.19% | 24,29,71 | 4.33% | 22,31,65 | 0.38% |
| 10,10 | 9,19 | 24,27,67 | 48.78 | 93.88% | 97.22% | 1.61% | 26,28,71 | 2.73% | 24,27,67 | 0.00% |
| 10,10 | 9,39 | 24,29,70 | 52.48 | 94.40% | 98.79% | 1.45% | 26,29,74 | 3.31% | 24,29,69 | 0.09% |
| 10,10 | 19,39 | 26,29,73 | 56.54 | 97.19% | 98.92% | 1.03% | 28,29,76 | 2.28% | 31,35,70 | 1.87% |

Table 4.3: Results of the supply chain configuration where $L_0 = 2$ and $L_1 = L_2 = 1$.

| scenario | | best base-stock policy | | | | | backorder model | | approximation | |
|---|---|---|---|---|---|---|---|---|---|---|
| $(\lambda_1, \lambda_2)$ | $(p_1, p_2)$ | $\boldsymbol{S}^*$ | $C(\boldsymbol{S}^*)$ | $\beta_1(\boldsymbol{S}^*)$ | $\beta_2(\boldsymbol{S}^*)$ | $CI(*)$ | $\boldsymbol{S}^{BO}$ | $CI(\boldsymbol{S}^{BO})$ | $\overline{\boldsymbol{S}}$ | $CI(\overline{\boldsymbol{S}})$ |
| 5,5 | 4,4 | 11,11,35 | 21.19 | 81.90% | 79.25% | 4.11% | 13,13,43 | 13.95% | 10,10,34 | 0.16% |
| 5,5 | 9,9 | 13,13,42 | 28.11 | 91.56% | 90.13% | 2.95% | 14,14,47 | 6.36% | 13,13,42 | 0.00% |
| 5,5 | 19,19 | 15,15,47 | 34.46 | 96.12% | 95.29% | 1.92% | 16,16,51 | 4.54% | 14,14,48 | 0.54% |
| 5,5 | 39,39 | 16,16,51 | 40.17 | 98.13% | 97.72% | 1.12% | 17,17,54 | 2.45% | 17,17,51 | 0.51% |
| 5,5 | 4,9 | 10,13,38 | 24.66 | 78.74% | 90.71% | 3.71% | 13,14,46 | 12.53% | 10,13,38 | 0.00% |
| 5,5 | 4,19 | 10,15,41 | 27.86 | 79.54% | 95.94% | 3.57% | 13,16,47 | 8.01% | 10,15,41 | 0.00% |
| 5,5 | 4,39 | 10,17,43 | 30.82 | 79.71% | 98.12% | 3.84% | 13,17,49 | 7.26% | 11,16,46 | 1.76% |
| 5,5 | 9,19 | 13,15,45 | 31.28 | 91.80% | 95.80% | 2.49% | 14,16,49 | 5.29% | 12,15,44 | 0.15% |
| 5,5 | 9,39 | 12,16,46 | 34.21 | 89.67% | 97.80% | 2.41% | 14,17,51 | 4.70% | 12,16,48 | 0.86% |
| 5,5 | 19,39 | 14,16,49 | 37.29 | 95.43% | 97.93% | 1.48% | 16,17,52 | 2.96% | 14,16,50 | 0.28% |
| 10,5 | 4,4 | 21,10,53 | 28.58 | 85.44% | 78.37% | 3.61% | 24,13,65 | 15.52% | 21,10,54 | 0.22% |
| 10,5 | 9,9 | 25,13,62 | 36.94 | 93.05% | 90.91% | 2.91% | 26,14,69 | 6.49% | 23,12,62 | 0.22% |
| 10,5 | 19,19 | 26,14,68 | 44.48 | 96.53% | 95.44% | 1.89% | 28,16,73 | 3.61% | 26,14,70 | 0.41% |
| 10,5 | 39,39 | 29,16,74 | 51.26 | 98.71% | 98.23% | 1.12% | 29,17,77 | 2.23% | 28,15,72 | 0.81% |
| 10,5 | 4,9 | 21,13,57 | 31.95 | 85.33% | 90.98% | 3.22% | 24,14,67 | 12.13% | 21,13,56 | 0.09% |
| 10,5 | 4,19 | 21,15,60 | 35.02 | 86.16% | 95.90% | 3.00% | 24,16,68 | 8.90% | 21,15,60 | 0.00% |
| 10,5 | 4,39 | 21,17,61 | 37.84 | 84.96% | 97.98% | 3.00% | 24,17,70 | 8.35% | 23,17,63 | 1.00% |
| 10,5 | 9,19 | 24,15,65 | 39.95 | 93.08% | 96.36% | 2.30% | 26,16,71 | 6.05% | 24,15,65 | 0.00% |
| 10,5 | 9,39 | 24,16,67 | 42.71 | 93.70% | 97.93% | 2.01% | 26,17,73 | 5.82% | 24,16,70 | 1.67% |
| 10,5 | 19,39 | 26,16,71 | 47.19 | 96.91% | 98.27% | 1.41% | 28,17,75 | 3.49% | 26,16,71 | 0.00% |
| 10,5 | 9,4 | 24,10,59 | 33.59 | 93.31% | 80.49% | 3.66% | 26,13,67 | 8.84% | 25,10,59 | 0.10% |
| 10,5 | 19,4 | 26,10,63 | 38.13 | 96.77% | 81.50% | 3.64% | 28,13,69 | 5.77% | 27,11,63 | 0.65% |
| 10,5 | 39,4 | 29,10,66 | 42.26 | 98.63% | 81.11% | 3.94% | 29,13,71 | 4.56% | 29,10,67 | 0.20% |
| 10,5 | 19,9 | 26,12,66 | 41.44 | 96.84% | 90.25% | 2.52% | 28,14,72 | 5.34% | 26,12,68 | 0.95% |
| 10,5 | 39,9 | 29,12,69 | 45.55 | 98.64% | 90.20% | 2.51% | 29,14,74 | 3.55% | 28,12,70 | 0.19% |
| 10,5 | 39,19 | 29,14,72 | 48.54 | 98.78% | 95.57% | 1.63% | 29,16,75 | 2.68% | 28,14,72 | 0.03% |
| 10,10 | 4,4 | 21,21,73 | 35.95 | 86.84% | 85.55% | 3.33% | 24,24,85 | 12.38% | 21,21,73 | 0.00% |
| 10,10 | 9,9 | 24,24,83 | 45.57 | 93.95% | 93.23% | 2.54% | 26,26,90 | 5.56% | 24,24,83 | 0.00% |
| 10,10 | 19,19 | 26,26,90 | 54.29 | 97.09% | 96.73% | 1.65% | 28,28,95 | 3.09% | 26,26,91 | 0.14% |
| 10,10 | 39,39 | 28,28,96 | 62.11 | 98.70% | 98.53% | 0.98% | 29,29,100 | 2.02% | 28,28,97 | 0.17% |
| 10,10 | 4,9 | 20,24,77 | 40.74 | 84.63% | 93.40% | 3.05% | 24,26,88 | 9.42% | 21,25,77 | 0.10% |
| 10,10 | 4,19 | 20,27,81 | 45.12 | 84.85% | 97.16% | 2.99% | 24,28,90 | 6.75% | 21,28,83 | 0.51% |
| 10,10 | 4,39 | 20,29,84 | 49.13 | 85.58% | 98.60% | 3.21% | 24,29,92 | 5.40% | 21,29,84 | 0.37% |
| 10,10 | 9,19 | 24,27,86 | 49.90 | 93.31% | 96.89% | 2.12% | 26,28,93 | 4.88% | 24,26,90 | 1.75% |
| 10,10 | 9,39 | 24,29,90 | 53.88 | 94.19% | 98.68% | 2.07% | 26,29,95 | 3.73% | 23,28,91 | 0.59% |
| 10,10 | 19,39 | 26,29,93 | 58.17 | 96.85% | 98.75% | 1.31% | 28,29,98 | 3.31% | 26,28,93 | 0.13% |

Table 4.4: Results of the supply chain configuration where $L_0 = L_1 = 2$ and $L_2 = 1$.

| scenario | | best base-stock policy | | | | | backorder model | | approximation | |
|---|---|---|---|---|---|---|---|---|---|---|
| $(\lambda_1, \lambda_2)$ | $(p_1, p_2)$ | $\boldsymbol{S}^*$ | $C(\boldsymbol{S}^*)$ | $\beta_1(\boldsymbol{S}^*)$ | $\beta_2(\boldsymbol{S}^*)$ | $CI(*)$ | $\boldsymbol{S}^{BO}$ | $CI(\boldsymbol{S}^{BO})$ | $\tilde{\boldsymbol{S}}$ | $CI(\tilde{\boldsymbol{S}})$ |
| 5,5 | 4,4 | 13,10,36 | 25.17 | 71.82% | 78.98% | 3.63% | 19,13,49 | 21.01% | 13,10,38 | 1.00% |
| 5,5 | 9,9 | 17,13,45 | 33.27 | 86.42% | 91.10% | 3.33% | 20,14,53 | 9.51% | 18,13,45 | 0.06% |
| 5,5 | 19,19 | 20,15,52 | 40.44 | 94.50% | 96.35% | 2.27% | 22,16,57 | 6.02% | 20,14,53 | 0.43% |
| 5,5 | 39,39 | 22,16,57 | 46.83 | 97.85% | 98.20% | - | 23,17,60 | 3.03% | 22,16,55 | 0.56% |
| 5,5 | 4,9 | 14,13,40 | 28.66 | 73.36% | 90.13% | 3.66% | 19,14,51 | 16.60% | 13,13,41 | 0.33% |
| 5,5 | 4,19 | 13,15,42 | 31.83 | 71.75% | 95.19% | 3.61% | 19,16,52 | 12.16% | 13,15,43 | 0.04% |
| 5,5 | 4,39 | 12,16,44 | 34.81 | 70.02% | 97.91% | 4.02% | 19,17,54 | 11.20% | 13,17,45 | 0.04% |
| 5,5 | 9,19 | 17,15,48 | 36.41 | 87.12% | 96.13% | 2.84% | 20,16,55 | 8.46% | 17,15,49 | 0.26% |
| 5,5 | 9,39 | 17,16,50 | 39.23 | 87.98% | 97.90% | 2.51% | 20,17,56 | 6.30% | 17,16,50 | 0.00% |
| 5,5 | 19,39 | 20,16,54 | 43.25 | 95.14% | 97.95% | - | 22,17,58 | 4.25% | 20,16,55 | 0.29% |
| 10,5 | 4,4 | 27,10,58 | 37.18 | 78.79% | 79.93% | - | 35,13,75 | 17.78% | 28,10,62 | 1.03% |
| 10,5 | 9,9 | 33,12,70 | 47.15 | 91.39% | 89.71% | - | 37,14,80 | 8.67% | 34,13,70 | 0.14% |
| 10,5 | 19,19 | 36,14,78 | 55.85 | 95.95% | 95.86% | - | 39,16,84 | 4.48% | 37,14,80 | 0.51% |
| 10,5 | 39,39 | 40,16,84 | 63.65 | 98.28% | 98.32% | - | 41,17,88 | 2.45% | 40,16,81 | 1.61% |
| 10,5 | 4,9 | 28,13,62 | 40.54 | 79.46% | 91.09% | - | 35,14,77 | 15.07% | 29,13,66 | 1.03% |
| 10,5 | 4,19 | 28,15,64 | 43.56 | 78.87% | 95.82% | - | 35,16,79 | 13.99% | 28,15,67 | 0.57% |
| 10,5 | 4,39 | 28,17,66 | 46.33 | 79.09% | 98.08% | - | 35,17,80 | 11.80% | 27,16,67 | 0.04% |
| 10,5 | 9,19 | 33,14,73 | 50.16 | 91.63% | 95.51% | - | 37,16,82 | 8.19% | 34,15,73 | 0.05% |
| 10,5 | 9,39 | 33,16,75 | 52.83 | 91.59% | 98.10% | - | 37,17,83 | 6.70% | 31,16,74 | 0.60% |
| 10,5 | 19,39 | 37,16,80 | 58.54 | 96.20% | 98.14% | - | 39,17,86 | 4.39% | 37,16,81 | 0.10% |
| 10,5 | 9,4 | 33,10,67 | 43.75 | 91.19% | 81.14% | - | 37,13,78 | 10.64% | 33,10,66 | 0.09% |
| 10,5 | 19,4 | 36,10,72 | 49.51 | 95.69% | 81.90% | - | 39,13,81 | 7.34% | 38,11,71 | 1.27% |
| 10,5 | 39,4 | 39,9,75 | 54.61 | 98.20% | 76.85% | - | 41,13,83 | 5.06% | 40,10,73 | 2.41% |
| 10,5 | 19,9 | 36,12,75 | 52.83 | 95.82% | 90.29% | - | 39,14,83 | 5.91% | 37,12,78 | 0.87% |
| 10,5 | 39,9 | 40,12,80 | 57.93 | 98.47% | 90.95% | - | 41,14,85 | 3.56% | 40,12,80 | 0.00% |
| 10,5 | 39,19 | 39,14,82 | 60.93 | 98.17% | 96.04% | - | 41,16,86 | 2.69% | 40,14,80 | 0.73% |
| 10,10 | 4,4 | 27,20,77 | 44.57 | 79.55% | 84.99% | - | 35,24,96 | 16.57% | 28,20,81 | 0.92% |
| 10,10 | 9,9 | 33,24,91 | 55.83 | 91.28% | 93.84% | - | 37,26,101 | 7.37% | 33,25,89 | 0.56% |
| 10,10 | 19,19 | 37,26,99 | 65.71 | 96.16% | 96.69% | - | 39,28,107 | 4.86% | 37,26,101 | 0.18% |
| 10,10 | 39,39 | 39,28,106 | 74.48 | 98.19% | 98.62% | - | 41,29,111 | 2.35% | 39,28,107 | 0.09% |
| 10,10 | 4,9 | 27,24,82 | 49.38 | 78.76% | 93.45% | - | 35,26,98 | 12.19% | 27,24,84 | 0.30% |
| 10,10 | 4,19 | 28,27,87 | 53.70 | 80.82% | 97.15% | - | 35,28,100 | 9.81% | 31,27,92 | 1.68% |
| 10,10 | 4,39 | 26,29,88 | 57.61 | 77.93% | 98.63% | - | 35,29,102 | 8.70% | 31,29,94 | 1.38% |
| 10,10 | 9,19 | 33,27,94 | 60.15 | 90.88% | 97.11% | - | 37,28,104 | 6.76% | 34,27,95 | 0.03% |
| 10,10 | 9,39 | 33,29,97 | 64.02 | 91.32% | 98.61% | - | 37,29,106 | 5.77% | 33,29,98 | 0.04% |
| 10,10 | 19,39 | 36,28,102 | 69.58 | 96.01% | 98.52% | - | 39,29,109 | 3.89% | 36,28,103 | 0.07% |

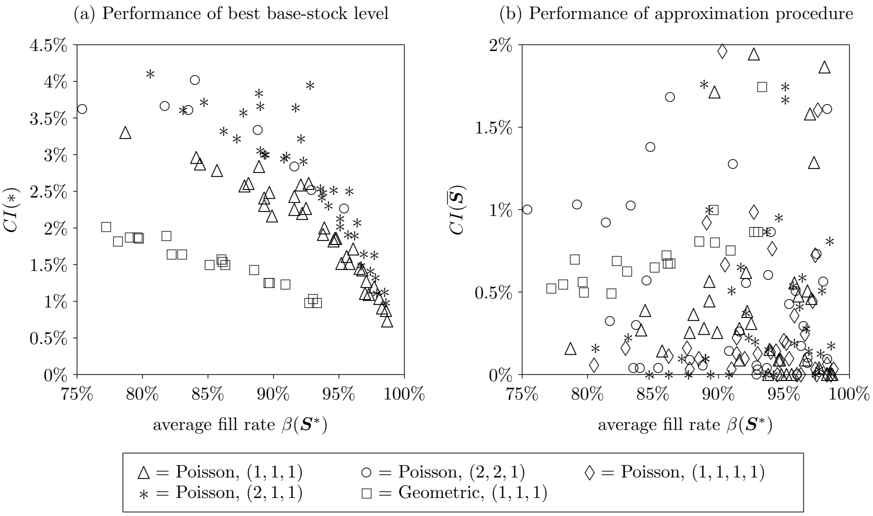


Figure 4.3: The relative cost performance of the best base-stock policy compared to the optimal policy (a) and the relative cost performance of the base-stock policy based on our approximation procedure compared to the the best base-stock policy (b) with on the x-axis the average fill rate over the retailer locations under the best base-stock policy ($\boldsymbol{S}^*$).

From these results, it is clear that echelon base-stock policies with linear rationing perform good with a cost increase of 1-2.5% compared to the optimal replenishment policy when the average fill rate at the retailers is 90% of higher. For moderate values of the fill rate (between 75-90%), these base-stock policies perform reasonably well with an average cost increase in the range of 2.5-4% compared to optimal replenishment policies. Overall, it can be concluded that base-stock policies perform closer to the optimal policy when the unit penalty cost increases (in comparison to the unit holding cost), which is consistent with the literature for single-echelon inventory systems with lost sales (Huh et al., 2009; Bijvank et al., 2014). When the demand follows a geometric distribution instead of a Poisson distribution, the best base-stock policies perform better (despite a higher variance-to-mean ratio). Increasing the lead time seems to reduce the effectiveness of base-stock policies, but still within acceptable ranges.

We have also tried to improve the best base-stock policy by adding a maximum order size for the retailers (as suggested in the modified base-stock policy by Bijvank & Johansen (2012) for the single-echelon inventory system). The results are found in Appendix 4.B.3, but it becomes clear that this modification of the base-stock policies does not have the same significant impact as compared to the single-echelon inventory system. The best base-stock policy performs on average around 0.2-0.6% better after the modification. One of the reasons for this is that the warehouse can have a shortage in case retailers order many units, which already results in less units being delivered to the retailers compared to their order quantities. In other words, the on-hand inventory available at the warehouse already provides an upper bound on the number of units that is sent to the retailers.

Furthermore, it is obvious that one cannot ignore the lost-sales assumption when determining the base-stock levels. Otherwise the base-stock levels are overestimated, which increases the average total cost by 4.5%, 6.3% and 8.5% (on average), respectively, based on the results in Table 4.2 to 4.4. This illustrates that the performance is worse when the lead time increases. Similar conclusions can be made when more retailers are included to the supply chain configuration (see Appendix 4.B.2). Finding the best base-stock policy through complete enumeration over all possible base-stock levels takes a lot of computational effort (exceeding 24 hours for certain instances). Therefore, it is important to have an efficient procedure that finds good base-stock policies. The resulting base-stock levels that minimize the average total cost based on our approximation procedure perform really well with a cost increase of less than 1% for most instances (regardless of the scenario under study). We have also analyzed the accuracy of the approximated average total cost itself, which is included in Appendix 4.B.4.

## 4.6. Conclusion

In this chapter, we consider one of the most fundamental supply chain networks in a retail setting: one warehouse with multiple sales locations. We focus on inventory replenishment policies when no backlogging is allowed at the retailers such that the demand of customers who arrive to a retailer with no on-hand inventories is lost. It is well known from the literature that the optimal replenishment policy for a simplified inventory system with just one retailer who is replenished by a supplier with ample inventory is already complex. Until now no work has ever studied optimal replenishment policies or the performance of echelon base-stock policies in a two-echelon divergent inventory system with lost sales. Our numerical results indicate that such replenishment policies perform mostly within 1-3% of the optimal policy. Finding the best base-stock levels is very challenging, if not impossible for real-sized inventory systems. Therefore, we propose a decomposition procedure where the probability distribution of the inventory level at a retail location is derived conditional on the echelon inventory level at the warehouse. Furthermore, the probability distribution of the echelon inventory level at the warehouse is found with the convolution of the satisfied demand over all retail retail locations as well as over the lead time. Expressions for the average satisfied demand at a retailer and its distribution are approximated conditional on the echelon inventory level at the warehouse. This results in an iterative procedure that is used to eventually approximate the total cost of the inventory system (consisting of holding and penalty costs). Our numerical results show that the approximation procedure can be used to find near-optimal base-stock levels (resulting in total costs that are within 1% of the total costs associated with the best base-stock policy).

A possible direction for future research can be the asymptotic optimality of echelon base-stock policies when the penalty cost grows large for divergent inventory systems with lost sales at the most downstream stock points. Our numerical results in Figure 4.3 clearly suggest that this seems to hold. However, finding a formal proof would be challenging since the optimal inventory allocation in case of a shortage at the upstream stock point is not straightforward. Another interesting research question is the impact of such shortage allocation rules as well as the impact of other replenishment policies. That being said, echelon base-stock policies with a simple allocation rule seem to perform really well already for moderate to high service levels (measured as the fill rate). Furthermore, our decomposition approach can be extended to more than two echelon inventory systems.

# Appendices

## 4.A. One-Period Holding Cost

The one-period holding cost in state $\xi_t$ can be rewritten as

$$\begin{aligned}
c(\xi_t) = {}& H_0 y_0 + \sum_{i=1}^{N} H_i \mathbb{E}\left[(y_i - D_i(t))^+\right] + \sum_{i=1}^{N} p_i \mathbb{E}\left[(D_i(t) - y_i)^+\right] \\
= {}& H_0 z_0 - \sum_{i=1}^{N} H_0 \mathbb{E}\left[z_i - D_i(t) + D_i(t)\right] + \sum_{i=1}^{N} H_i \mathbb{E}\left[(z_i - D_i(t))^+\right] \\
& + \sum_{i=1}^{N} p_i \mathbb{E}\left[(D_i(t) - z_i)^+\right] \\
= {}& H_0 z_0 - \sum_{i=1}^{N} H_0 \mathbb{E}\left[z_i - D_i(t) + (D_i(t) - z_i)^+\right] \\
& - \sum_{i=1}^{N} H_0 \mathbb{E}\left[D_i(t) - (D_i(t) - z_i)^+\right] + \sum_{i=1}^{N} H_i \mathbb{E}\left[(z_i - D_i(t))^+\right] \\
& + \sum_{i=1}^{N} p_i \mathbb{E}\left[(D_i(t) - z_i)^+\right] \\
= {}& H_0 z_0 - \sum_{i=1}^{N} H_0 \mathbb{E}\left[(z_i - D_i(t))^+\right] - \sum_{i=1}^{N} H_0 \mathbb{E}\left[\min\left\{D_i(t), z_i\right\}\right] \\
& + \sum_{i=1}^{N} H_i \mathbb{E}\left[(z_i - D_i(t))^+\right] + \sum_{i=1}^{N} p_i \mathbb{E}\left[(D_i(t) - z_i)^+\right] \\
= {}& H_0 \left( z_0 - \sum_{i=1}^{N} \mathbb{E}\left[\min\{D_i(t), z_i\}\right] \right) + \sum_{i=1}^{N} (H_i - H_0)\, \mathbb{E}\left[(z_i - D_i(t))^+\right] \\
& + \sum_{i=1}^{N} p_i \mathbb{E}\left[(D_i(t) - z_i)^+\right]
\end{aligned}$$

# 4.B. Additional Numerical Results

## 4.B.1 Geometric Demand Distribution

When the demand follows a geometric distribution and the supply chain configuration is the same as in Table 4.2, the results are given by Table 4.B.1. When comparing the results from both tables, the relative cost increase for using echelon base-stock policies (i.e., $CI(*)$) as well as the relative cost increase for using our approximation procedure (i.e., $CI(\overline{\boldsymbol{S}})$) is very similar.

Table 4.B.1: Results of the supply chain configuration where $L_0 = 1$ and $L_1 = L_2 = 1$ with geometrically distributed demand.

| scenario | | best base-stock policy | | | | | backorder model | | approximation | |
|---|---|---|---|---|---|---|---|---|---|---|
| $(\lambda_1, \lambda_2)$ | $(p_1, p_2)$ | $\boldsymbol{S}^*$ | $C(\boldsymbol{S}^*)$ | $\beta_1(\boldsymbol{S}^*)$ | $\beta_2(\boldsymbol{S}^*)$ | $CI(*)$ | $\boldsymbol{S}^{BO}$ | $CI(\boldsymbol{S}^{BO})$ | $\overline{\boldsymbol{S}}$ | $CI(\overline{\boldsymbol{S}})$ |
| 5,5 | 4,4 | 8,8,18 | 32.16 | 49.56% | 47.52% | 2.28% | 13,13,33 | 19.44% | 9,9,18 | 0.17% |
| 5,5 | 9,9 | 14,14,31 | 51.15 | 72.43% | 71.24% | 2.14% | 14,14,37 | 3.86% | 15,15,30 | 0.47% |
| 5,5 | 19,19 | 19,19,43 | 70.94 | 85.42% | 84.85% | 1.51% | 16,16,39 | 2.35% | 21,21,43 | 0.65% |
| 5,5 | 39,39 | 24,24,55 | 90.79 | 92.80% | 92.55% | 0.99% | 17,17,42 | 15.55% | 24,24,52 | 0.87% |
| 5,5 | 4,9 | 8,14,25 | 41.73 | 50.36% | 71.83% | 2.39% | 13,14,35 | 9.97% | 8,16,24 | 0.34% |
| 5,5 | 4,19 | 8,20,31 | 51.87 | 50.58% | 85.05% | 2.28% | 13,16,36 | 6.99% | 8,22,30 | 0.47% |
| 5,5 | 4,39 | 8,26,38 | 62.18 | 52.43% | 93.04% | 2.08% | 13,17,38 | 15.76% | 8,28,36 | 0.51% |
| 5,5 | 9,19 | 14,20,38 | 61.12 | 73.41% | 85.84% | 1.87% | 14,16,38 | 2.35% | 15,22,37 | 0.56% |
| 5,5 | 9,39 | 13,25,43 | 71.23 | 71.87% | 92.59% | 1.65% | 14,17,40 | 10.30% | 15,28,43 | 0.69% |
| 5,5 | 19,39 | 19,25,50 | 80.91 | 86.05% | 93.07% | 1.25% | 16,17,41 | 9.83% | 22,27,52 | 1.00% |
| 10,5 | 4,4 | 16,8,27 | 47.38 | 49.29% | 49.01% | 2.36% | 24,13,49 | 18.90% | 18,9,27 | 0.14% |
| 10,5 | 9,9 | 28,14,47 | 75.20 | 73.09% | 72.86% | 2.23% | 26,14,53 | 2.86% | 30,15,46 | 0.33% |
| 10,5 | 19,19 | 38,19,65 | 104.19 | 86.05% | 85.86% | 1.57% | 28,16,57 | 4.92% | 42,22,64 | 0.72% |
| 10,5 | 39,39 | 48,24,82 | 133.32 | 93.03% | 92.91% | 1.03% | 29,17,60 | 23.30% | 52,27,81 | 0.87% |
| 10,5 | 4,9 | 16,14,34 | 56.81 | 50.37% | 72.17% | 2.37% | 24,14,51 | 12.07% | 17,16,33 | 0.31% |
| 10,5 | 4,19 | 15,20,40 | 66.74 | 49.54% | 85.95% | 2.31% | 24,16,52 | 9.06% | 17,23,40 | 0.48% |
| 10,5 | 4,39 | 15,25,46 | 76.83 | 50.77% | 92.59% | 2.30% | 24,17,54 | 15.62% | 17,29,46 | 0.54% |
| 10,5 | 9,19 | 27,20,53 | 84.95 | 72.66% | 86.36% | 2.02% | 26,16,55 | 2.58% | 30,22,52 | 0.52% |
| 10,5 | 9,39 | 26,25,58 | 94.84 | 72.01% | 92.91% | 1.88% | 26,17,56 | 8.31% | 29,28,58 | 0.70% |
| 10,5 | 19,39 | 38,25,71 | 113.96 | 86.19% | 93.17% | 1.43% | 28,17,58 | 9.65% | 42,28,70 | 0.81% |
| 10,5 | 9,4 | 28,8,40 | 66.00 | 72.15% | 50.30% | 2.57% | 26,13,51 | 6.30% | 31,9,40 | 0.27% |
| 10,5 | 19,4 | 38,8,53 | 85.70 | 85.63% | 52.54% | 2.27% | 28,13,53 | 8.06% | 43,9,52 | 0.40% |
| 10,5 | 39,4 | 48,7,64 | 105.55 | 92.85% | 48.63% | 1.82% | 29,13,55 | 25.79% | 55,9,64 | 0.55% |
| 10,5 | 19,9 | 38,14,59 | 94.67 | 85.52% | 74.44% | 1.89% | 28,14,56 | 5.28% | 43,15,58 | 0.49% |
| 10,5 | 39,9 | 49,13,70 | 114.37 | 92.85% | 72.57% | 1.54% | 29,14,57 | 21.47% | 55,16,71 | 0.68% |
| 10,5 | 39,19 | 49,19,77 | 123.74 | 93.17% | 86.46% | 1.23% | 29,16,59 | 20.36% | 53,21,75 | 0.75% |
| 10,10 | 4,4 | 16,16,36 | 62.52 | 50.15% | 49.01% | 2.34% | 24,24,64 | 17.50% | 17,18,35 | 0.14% |
| 10,10 | 9,9 | 27,28,62 | 98.84 | 72.51% | 73.58% | 2.15% | 26,26,69 | 2.70% | 31,31,62 | 0.38% |
| 10,10 | 19,19 | 38,38,86 | 136.65 | 86.41% | 86.11% | 1.50% | 28,28,73 | 6.51% | 43,43,86 | 0.67% |
| 10,10 | 39,39 | 48,48,109 | 174.60 | 93.38% | 93.24% | 0.99% | 29,29,77 | 28.15% | 48,48,100 | 1.75% |
| 10,10 | 4,9 | 16,28,49 | 80.81 | 50.54% | 72.40% | 2.39% | 24,26,67 | 8.73% | 18,31,49 | 0.25% |
| 10,10 | 4,19 | 16,39,62 | 100.17 | 52.34% | 85.84% | 2.26% | 24,28,69 | 9.53% | 17,44,61 | 0.39% |
| 10,10 | 4,39 | 16,50,74 | 119.80 | 53.44% | 92.99% | 2.05% | 24,29,71 | 24.53% | 16,53,73 | 0.25% |
| 10,10 | 9,19 | 27,38,74 | 117.87 | 73.74% | 85.76% | 1.86% | 26,28,71 | 4.78% | 30,43,74 | 0.50% |
| 10,10 | 9,39 | 26,49,85 | 137.19 | 73.08% | 92.99% | 1.65% | 26,29,74 | 18.81% | 30,55,85 | 0.63% |
| 10,10 | 19,39 | 37,48,97 | 155.72 | 86.29% | 93.01% | 1.25% | 28,29,76 | 18.71% | 42,55,97 | 0.80% |

### 4.B.2 Supply Chain with Three Retailers

When the supply chain consists of three retailers instead of two and the supply chain configuration is similar to Table 4.2 (i.e., all lead times are one review period), the results are given by Table 4.B.2. For these instances, no optimal replenishment policy can be found within at least 24 hours of computation time. When comparing the results from both tables, the base-stock levels $\overline{\boldsymbol{S}}$ found with our approximation procedure result in even a better performance compared to the supply chain configuration with only two retailer (i.e., comparing $CI(\overline{\boldsymbol{S}})$ from both tables).

Table 4.B.2: Results of the supply chain configuration with three retailers where $L_0 = 1$ and $L_1 = L_2 = L_3 = 1$.

| scenario | | best base-stock policy | | | | | backorder model | | approximation | |
|---|---|---|---|---|---|---|---|---|---|---|
| $(\lambda_1,\lambda_2,\lambda_3)$ | $(p_1,p_2,p_3)$ | $\boldsymbol{S}^*$ | $C(\boldsymbol{S}^*)$ | $\beta_1(\boldsymbol{S}^*)$ | $\beta_2(\boldsymbol{S}^*)$ | $\beta_3(\boldsymbol{S}^*)$ | $\boldsymbol{S}^{BO}$ | $CI(\boldsymbol{S}^{BO})$ | $\overline{\boldsymbol{S}}$ | $CI(\overline{\boldsymbol{S}})$ |
| 5,5,5 | 4,4,4 | 11,11,10,40 | 31.26 | 82.44% | 80.60% | 78.46% | 13,13,13,49 | 10.27% | 13,13,13,40 | 0.05% |
| 5,5,5 | 9,9,9 | 13,13,13,49 | 40.94 | 92.41% | 91.42% | 90.29% | 14,14,14,54 | 4.09% | 13,13,13,48 | 0.12% |
| 5,5,5 | 19,19,19 | 14,15,15,55 | 49.67 | 95.13% | 96.49% | 95.92% | 16,16,16,59 | 2.59% | 15,15,15,54 | 0.36% |
| 5,5,5 | 39,39,39 | 16,16,16,61 | 57.48 | 98.55% | 98.32% | 98.04% | 17,17,17,63 | 1.35% | 16,16,16,61 | 0.00% |
| 5,5,5 | 4,9,9 | 10,13,13,46 | 37.69 | 79.82% | 92.09% | 91.02% | 13,14,14,53 | 7.29% | 12,15,15,46 | 0.16% |
| 5,5,5 | 4,19,19 | 10,15,15,50 | 43.53 | 79.40% | 96.38% | 95.74% | 13,16,16,56 | 4.93% | 13,17,18,51 | 0.67% |
| 5,5,5 | 4,39,39 | 10,16,16,54 | 48.82 | 81.77% | 98.26% | 98.02% | 13,17,17,58 | 3.03% | 10,16,16,52 | 0.99% |
| 5,5,5 | 9,19,19 | 13,15,15,53 | 46.73 | 91.84% | 96.08% | 95.44% | 14,16,16,58 | 3.93% | 13,15,15,53 | 0.00% |
| 5,5,5 | 9,39,39 | 12,16,16,56 | 51.98 | 90.14% | 98.21% | 97.89% | 14,17,17,60 | 1.99% | 13,17,17,57 | 0.09% |
| 5,5,5 | 19,39,39 | 14,16,16,59 | 54.83 | 96.00% | 98.37% | 98.09% | 16,17,17,62 | 2.31% | 14,16,16,59 | 0.00% |
| 10,5,5 | 4,4,4 | 21,10,10,54 | 38.57 | 86.47% | 80.07% | 78.67% | 24,13,13,65 | 10.12% | 24,12,12,55 | 0.16% |
| 10,5,5 | 9,9,9 | 25,13,13,65 | 49.54 | 94.09% | 92.21% | 91.43% | 26,14,14,71 | 4.63% | 27,14,14,64 | 0.13% |
| 10,5,5 | 19,19,19 | 26,14,14,72 | 59.33 | 97.18% | 96.22% | 95.79% | 28,16,16,76 | 2.52% | 29,16,16,71 | 0.25% |
| 10,5,5 | 39,39,39 | 29,16,16,78 | 68.07 | 98.82% | 98.42% | 98.15% | 29,17,17,80 | 1.15% | 29,16,16,78 | 0.00% |
| 10,5,5 | 4,9,9 | 21,13,13,60 | 44.96 | 85.51% | 92.05% | 90.76% | 24,14,14,68 | 5.86% | 23,14,14,60 | 0.09% |
| 10,5,5 | 4,19,19 | 21,15,15,65 | 50.72 | 86.57% | 96.65% | 96.00% | 24,16,16,72 | 5.62% | 21,15,15,66 | 0.22% |
| 10,5,5 | 4,39,39 | 20,16,16,68 | 55.93 | 85.70% | 98.40% | 98.14% | 24,17,17,74 | 3.63% | 21,16,16,68 | 0.10% |
| 10,5,5 | 9,19,19 | 25,15,15,69 | 55.25 | 94.20% | 96.46% | 95.80% | 26,16,16,74 | 3.51% | 27,16,16,69 | 0.19% |
| 10,5,5 | 9,39,39 | 24,16,16,72 | 60.40 | 93.98% | 98.22% | 97.95% | 26,17,17,77 | 2.90% | 24,16,16,72 | 0.00% |
| 10,5,5 | 19,39,39 | 26,16,16,75 | 64.47 | 96.84% | 98.44% | 98.13% | 28,17,17,79 | 2.00% | 26,17,17,73 | 1.60% |
| 10,5,5 | 9,4,4 | 28,12,12,59 | 43.24 | 93.45% | 82.92% | 81.35% | 26,13,13,67 | 6.78% | 29,12,12,58 | 0.03% |
| 10,5,5 | 19,4,4 | 27,10,10,62 | 47.25 | 97.28% | 81.80% | 80.11% | 28,13,13,69 | 5.64% | 28,11,12,63 | 0.92% |
| 10,5,5 | 39,4,4 | 29,10,10,65 | 51.04 | 98.79% | 82.65% | 80.93% | 29,13,13,71 | 5.62% | 27,11,11,67 | 1.96% |
| 10,5,5 | 19,9,9 | 27,13,13,68 | 53.63 | 97.05% | 92.61% | 91.90% | 28,14,14,73 | 3.41% | 28,13,13,68 | 0.09% |
| 10,5,5 | 39,9,9 | 29,13,12,70 | 57.35 | 98.68% | 92.55% | 90.24% | 29,14,14,75 | 2.88% | 30,13,13,70 | 0.20% |
| 10,5,5 | 39,19,19 | 29,14,14,72 | 63.47 | 98.85% | 95.79% | 95.37% | 29,16,16,78 | 1.85% | 34,16,16,72 | 0.45% |
| 10,10,10 | 4,4,4 | 21,21,21,83 | 53.14 | 87.14% | 86.23% | 85.25% | 24,24,24,96 | 8.45% | 23,23,23,84 | 0.11% |
| 10,10,10 | 9,9,9 | 24,24,24,96 | 66.55 | 94.55% | 94.07% | 93.53% | 26,26,26,103 | 3.69% | 25,25,25,99 | 0.76% |
| 10,10,10 | 19,19,19 | 27,27,26,105 | 78.48 | 97.66% | 97.40% | 97.02% | 28,28,28,109 | 1.61% | 27,27,27,102 | 0.72% |
| 10,10,10 | 39,39,39 | 28,28,28,112 | 89.14 | 98.88% | 98.76% | 98.62% | 29,29,29,115 | 0.79% | 29,29,29,112 | 0.03% |
| 10,10,10 | 4,9,9 | 20,24,24,91 | 62.06 | 85.37% | 94.11% | 93.57% | 24,26,26,101 | 5.44% | 21,25,25,91 | 0.03% |
| 10,10,10 | 4,19,19 | 21,27,27,98 | 70.05 | 87.54% | 97.40% | 97.13% | 24,28,28,105 | 3.59% | 20,27,27,96 | 0.13% |
| 10,10,10 | 4,39,39 | 20,29,29,102 | 77.22 | 85.76% | 98.85% | 98.68% | 24,29,29,109 | 3.00% | 20,29,29,101 | 0.15% |
| 10,10,10 | 9,19,19 | 24,27,27,102 | 74.46 | 94.14% | 97.51% | 97.22% | 26,28,28,107 | 2.28% | 24,27,27,102 | 0.00% |
| 10,10,10 | 9,39,39 | 24,29,29,107 | 81.59 | 94.42% | 98.90% | 98.75% | 26,29,29,111 | 1.90% | 24,29,29,106 | 0.03% |
| 10,10,10 | 19,39,39 | 26,29,29,110 | 85.58 | 97.17% | 98.98% | 98.85% | 28,29,29,114 | 1.89% | 27,29,29,110 | 0.07% |

### 4.B.3 Modified Base-Stock Policy

Bijvank & Johansen (2012) suggest to include an upper bound $\overline{q}$ on the order quantities in a single-echelon inventory system where excess demand is lost and a base-stock policy is used. The authors propose to set this maximum order quantity by

$$\overline{q}_i = \lfloor S_i/(L_i + R) \rfloor \,, \quad i = 1, \ldots, N$$

where $\lfloor x \rfloor$ equals $x$ rounded down to the nearest integer.

When we apply these upper bounds to the best base-stock policies $\boldsymbol{S}^*$ in our multi-echelon setting, we obtain the relative cost increases compared the standard best echelon base-stock policies as reported in Table 4.B.3. Even though there is no significant impact, it is interesting to note that the modified base-stock policies improve the regular base-stock policies when demand follows a Poisson distribution (they have a negative cost increase), whereas the modified base-stock policies perform worse when demand follows a geometric distribution.

Table 4.B.3: The relative cost increase of the modified base-stock policy applied to $\boldsymbol{S}^*$.

| scenario | | supply chain configuration $(L_0, L_1, L_2)$ | | | | |
|---|---|---|---|---|---|---|
| $(\lambda_1, \lambda_2)$ | $(p_1, p_2)$ | $(1,1,1)$ | $(2,1,1)$ | $(2,2,1)$ | $(1,1,1,1)$ | $(1,1,1)$ geo. $D_i$ |
| 5,5 | 4,4 | -1.04% | -1.04% | -0.20% | -1.45% | -0.28% |
| 5,5 | 9,9 | -0.73% | -0.71% | -0.57% | -0.94% | -0.26% |
| 5,5 | 19,19 | -0.37% | -0.34% | -0.17% | -0.50% | -0.13% |
| 5,5 | 39,39 | -0.11% | -0.05% | 0.07% | -0.16% | 0.06% |
| 5,5 | 4,9 | -1.02% | -0.95% | -1.05% | -1.06% | -0.23% |
| 5,5 | 4,19 | -0.61% | -0.60% | -0.35% | -0.74% | -0.16% |
| 5,5 | 4,39 | -0.70% | -0.42% | -0.63% | -0.57% | -0.09% |
| 5,5 | 9,19 | -0.60% | -0.52% | -0.21% | -0.69% | -0.15% |
| 5,5 | 9,39 | -0.37% | -0.50% | -0.68% | -0.44% | -0.11% |
| 5,5 | 19,39 | -0.28% | -0.24% | -0.50% | -0.30% | -0.06% |
| 10,5 | 4,4 | -0.99% | -0.92% | -0.79% | -1.30% | -0.27% |
| 10,5 | 9,9 | -0.67% | -0.63% | -0.68% | -0.87% | -0.22% |
| 10,5 | 19,19 | -0.33% | -0.32% | -0.18% | -0.45% | -0.07% |
| 10,5 | 39,39 | -0.16% | -0.13% | 0.26% | -0.21% | 0.06% |
| 10,5 | 4,9 | -0.90% | -0.88% | -0.60% | -1.02% | -0.31% |
| 10,5 | 4,19 | -0.71% | -0.63% | 0.32% | -0.68% | -0.20% |
| 10,5 | 4,39 | -0.29% | -0.43% | 0.38% | -0.36% | -0.21% |
| 10,5 | 9,19 | -0.29% | -0.26% | -0.47% | -0.61% | -0.18% |
| 10,5 | 9,39 | -0.46% | -0.41% | -0.29% | -0.46% | -0.16% |
| 10,5 | 19,39 | -0.21% | -0.20% | -0.04% | -0.22% | -0.06% |
| 10,5 | 9,4 | -0.82% | -0.69% | -0.47% | -0.79% | -0.16% |
| 10,5 | 19,4 | -0.43% | -0.47% | -0.23% | -0.84% | -0.10% |
| 10,5 | 39,4 | -0.63% | -0.57% | 0.22% | -0.75% | 0.06% |
| 10,5 | 19,9 | -0.51% | -0.46% | -0.37% | -0.70% | -0.11% |
| 10,5 | 39,9 | -0.47% | -0.33% | -0.02% | -0.62% | -0.08% |
| 10,5 | 39,19 | -0.24% | -0.23% | -0.11% | -0.48% | -0.02% |
| 10,10 | 4,4 | -0.95% | -0.91% | -0.76% | -1.27% | -0.40% |
| 10,10 | 9,9 | -0.63% | -0.57% | -0.58% | -0.79% | -0.32% |
| 10,10 | 19,19 | -0.34% | -0.31% | -0.26% | -0.44% | -0.09% |
| 10,10 | 39,39 | -0.15% | -0.12% | -0.12% | -0.21% | 0.04% |
| 10,10 | 4,9 | -0.77% | -0.68% | -0.50% | -0.93% | -0.28% |
| 10,10 | 4,19 | -0.41% | -0.45% | 0.22% | -0.74% | -0.18% |
| 10,10 | 4,39 | -0.51% | -0.49% | -0.37% | -0.43% | -0.10% |
| 10,10 | 9,19 | -0.52% | -0.52% | -0.12% | -0.58% | -0.17% |
| 10,10 | 9,39 | -0.44% | -0.47% | -0.30% | -0.41% | -0.11% |
| 10,10 | 19,39 | -0.21% | -0.21% | -0.30% | -0.26% | -0.03% |

### 4.B.4 Accuracy Approximation Procedure

Let us denote the approximated average total cost given by Eq. (4.5) by $\hat{C}(\boldsymbol{S})$ for a given echelon base-stock policy $\boldsymbol{S}$. Table 4.B.4 reports both the approximated total cost $\hat{C}(\overline{\boldsymbol{S}})$ as well as the actual total cost $C(\overline{\boldsymbol{S}})$ for the echelon base-stock policies $\overline{\boldsymbol{S}}$ (i.e., the base-stock policies that minimize $\hat{C}(\overline{\boldsymbol{S}})$). From these results we conclude that the approximation procedure results in similar average total cost.

Table 4.B.4: The approximated total cost $\hat{C}(\overline{\boldsymbol{S}})$ and actual total cost $C(\overline{\boldsymbol{S}})$ for echelon base-stock policies $\overline{\boldsymbol{S}}$.

| scenario | | supply chain configuration $(L_0, L_1, L_2)$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $(1,1,1)$ | | $(2,1,1)$ | | $(2,2,1)$ | | $(1,1,1,1)$ | | $(1,1,1)$ geo. $D_i$ | |
| $(\lambda_1,\lambda_2)$ | $(p_1,p_2)$ | $C(\overline{\boldsymbol{S}})$ | $\hat{C}(\overline{\boldsymbol{S}})$ | $C(\overline{\boldsymbol{S}})$ | $\hat{C}(\overline{\boldsymbol{S}})$ | $C(\overline{\boldsymbol{S}})$ | $\hat{C}(\overline{\boldsymbol{S}})$ | $C(\overline{\boldsymbol{S}})$ | $\hat{C}(\overline{\boldsymbol{S}})$ | $C(\overline{\boldsymbol{S}})$ | $\hat{C}(\overline{\boldsymbol{S}})$ |
| 5,5 | 4,4 | 20.98 | 20.80 | 21.22 | 21.56 | 25.43 | 25.83 | 31.28 | 31.09 | 32.21 | 32.01 |
| 5,5 | 9,9 | 27.63 | 27.47 | 28.11 | 28.67 | 33.29 | 33.61 | 41.00 | 40.75 | 51.39 | 51.01 |
| 5,5 | 19,19 | 33.69 | 33.31 | 34.65 | 34.80 | 40.62 | 40.68 | 49.85 | 49.19 | 71.40 | 70.78 |
| 5,5 | 39,39 | 38.84 | 38.79 | 40.38 | 40.23 | 47.09 | 46.80 | 57.48 | 57.25 | 91.58 | 90.48 |
| 5,5 | 4,9 | 24.34 | 24.23 | 24.66 | 25.15 | 28.76 | 29.03 | 37.75 | 37.63 | 41.87 | 41.59 |
| 5,5 | 4,19 | 27.35 | 27.32 | 27.86 | 28.51 | 31.84 | 32.20 | 43.82 | 42.98 | 52.12 | 51.73 |
| 5,5 | 4,39 | 30.49 | 29.52 | 31.37 | 31.54 | 34.82 | 35.23 | 49.31 | 46.98 | 62.49 | 62.08 |
| 5,5 | 9,19 | 30.51 | 30.33 | 31.33 | 31.87 | 36.51 | 36.70 | 46.73 | 46.29 | 61.46 | 61.02 |
| 5,5 | 9,39 | 33.28 | 32.86 | 34.50 | 34.61 | 39.23 | 39.01 | 52.03 | 51.11 | 71.72 | 71.12 |
| 5,5 | 19,39 | 36.71 | 36.23 | 37.39 | 37.37 | 43.38 | 43.16 | 54.83 | 54.76 | 81.72 | 80.62 |
| 10,5 | 4,4 | 28.35 | 28.23 | 28.64 | 29.20 | 37.56 | 38.19 | 38.63 | 38.46 | 47.44 | 47.18 |
| 10,5 | 9,9 | 36.31 | 36.29 | 37.03 | 37.67 | 47.21 | 47.71 | 49.60 | 49.52 | 75.44 | 74.95 |
| 10,5 | 19,19 | 43.47 | 42.91 | 44.66 | 44.76 | 56.13 | 56.02 | 59.48 | 58.66 | 104.94 | 103.86 |
| 10,5 | 39,39 | 49.58 | 49.58 | 51.68 | 51.50 | 64.68 | 64.15 | 68.07 | 67.87 | 134.47 | 132.40 |
| 10,5 | 4,9 | 31.59 | 31.49 | 31.98 | 32.56 | 40.95 | 41.36 | 45.00 | 44.89 | 56.99 | 56.68 |
| 10,5 | 4,19 | 34.64 | 34.26 | 35.02 | 35.77 | 43.81 | 44.20 | 50.83 | 50.02 | 67.06 | 66.75 |
| 10,5 | 4,39 | 37.21 | 36.85 | 38.21 | 38.37 | 46.35 | 46.85 | 55.98 | 54.22 | 77.24 | 76.97 |
| 10,5 | 9,19 | 39.10 | 38.89 | 39.95 | 40.89 | 50.18 | 50.80 | 55.36 | 54.74 | 85.40 | 84.84 |
| 10,5 | 9,39 | 41.70 | 41.23 | 43.43 | 43.43 | 53.15 | 53.36 | 60.40 | 59.38 | 95.50 | 94.89 |
| 10,5 | 19,39 | 46.42 | 45.78 | 47.19 | 47.19 | 58.60 | 58.30 | 65.50 | 64.31 | 114.88 | 113.45 |
| 10,5 | 9,4 | 33.08 | 33.06 | 33.62 | 34.45 | 43.78 | 44.45 | 43.26 | 43.14 | 66.18 | 65.66 |
| 10,5 | 19,4 | 37.41 | 36.64 | 38.38 | 38.75 | 50.15 | 50.28 | 47.68 | 46.92 | 86.04 | 85.25 |
| 10,5 | 39,4 | 41.79 | 39.66 | 42.34 | 42.57 | 55.93 | 52.02 | 52.04 | 50.35 | 106.13 | 105.19 |
| 10,5 | 19,9 | 40.38 | 40.03 | 41.83 | 42.10 | 53.29 | 53.42 | 53.67 | 53.11 | 95.13 | 94.24 |
| 10,5 | 39,9 | 44.32 | 43.43 | 45.63 | 45.62 | 57.93 | 57.57 | 57.46 | 56.40 | 115.14 | 113.60 |
| 10,5 | 39,19 | 47.00 | 47.06 | 48.55 | 48.37 | 61.37 | 61.01 | 63.75 | 62.46 | 124.67 | 122.86 |
| 10,10 | 4,4 | 35.65 | 35.50 | 35.95 | 36.66 | 44.98 | 45.64 | 53.19 | 53.04 | 62.60 | 62.33 |
| 10,10 | 9,9 | 44.79 | 44.82 | 45.57 | 46.57 | 56.14 | 56.60 | 67.05 | 66.49 | 99.22 | 98.65 |
| 10,10 | 19,19 | 53.12 | 52.41 | 54.36 | 54.33 | 65.83 | 65.56 | 79.04 | 77.68 | 137.57 | 136.46 |
| 10,10 | 39,39 | 60.17 | 60.09 | 62.22 | 62.73 | 74.55 | 75.13 | 89.17 | 88.72 | 177.65 | 174.82 |
| 10,10 | 4,9 | 40.33 | 40.07 | 40.78 | 41.65 | 49.53 | 50.10 | 62.08 | 61.60 | 81.01 | 80.62 |
| 10,10 | 4,19 | 44.33 | 43.51 | 45.35 | 45.64 | 54.60 | 54.44 | 70.14 | 68.43 | 100.56 | 100.00 |
| 10,10 | 4,39 | 48.19 | 46.65 | 49.31 | 49.07 | 58.40 | 57.70 | 77.33 | 74.14 | 120.10 | 119.81 |
| 10,10 | 9,19 | 48.78 | 48.33 | 50.77 | 50.80 | 60.17 | 61.02 | 74.46 | 73.45 | 118.46 | 117.80 |
| 10,10 | 9,39 | 52.52 | 51.62 | 54.20 | 54.05 | 64.04 | 64.82 | 81.62 | 79.71 | 138.05 | 137.08 |
| 10,10 | 19,39 | 57.59 | 56.28 | 58.25 | 57.91 | 69.63 | 68.97 | 85.64 | 85.33 | 156.97 | 155.20 |